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COUPLING BETWEEN RECTANGULAR OPTICAL WAVEGUIDES

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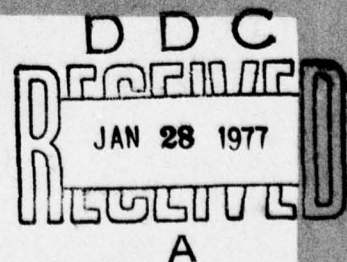Chicago, Illinois

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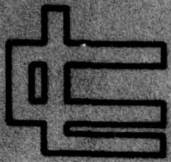
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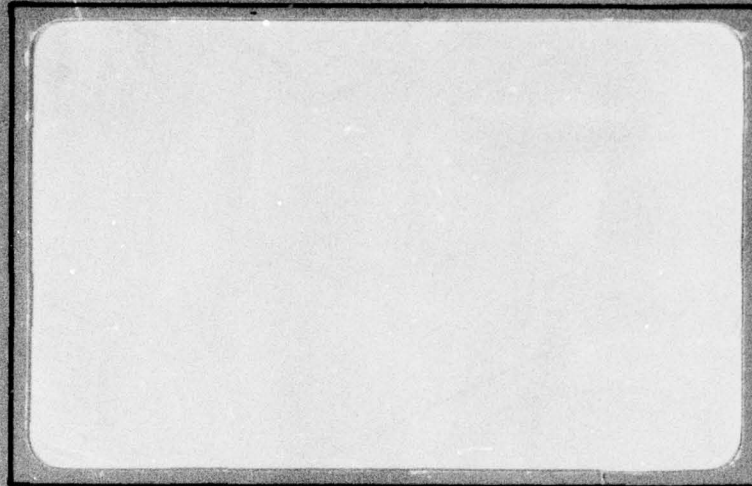
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A.-G. Kazkaz and P.L.E. Uslenghi

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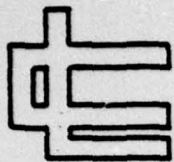| REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
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A theory is developed for strong coupling between two rectangular optical waveguides, that contains the Miller-Marcatili result as a limiting case. The theory allows for a more accurate design of a directional coupler than was previously possible.
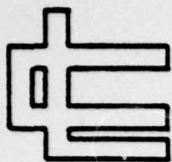
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PREFACE

This report was written by the first author under the guidance and supervision of the second author. It represents Mr. Kazkaz' thesis for the degree of Master of Science in Information Engineering.

December 1976.   Piergiorgio L.E. Uslenghi
                                         Director of the Laboratory

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INTRODUCTION

In modern information engineering, new methods have been found for the the transmission of data; among these are optical waveguides and fibers. The waveguides that are capable of guiding beams of light can be made in substrates as integrated optical circuits capable of performances analogous to those of integrated electronic circuits for digital operations, or can be made as round optical fibers capable of performance similar to that of coaxial cables in communications.

Marcatili [1] studied a rectangular waveguide of dielectric material immersed in a dielectric substrate of refractive index smaller than that of the guide (Fig. 1-a). He showed that two kinds of modes can be carried by the guide. The number of modes and their cutoff frequencies depend on the dimensions of the guide, its refractive index and the refractive index of the surrounding medium.

To transfer energy from one rectangular waveguide into another, the two guides are made parallel to each other, as shown in Fig. 1-b. In the case of identical waveguides, the coupling between them depends on the distance separating them and the distance they have in common along their axis. To control the transfer of power from one guide to the other, we have to study the effect of these two parameters.

Miller [2] studied the coupling between transmission lines. Marcatili adopted his solution and found the coupling coefficients for the two identical rectangular waveguides [3]. However, Miller used approximations which are valid only when the separation distance between the two guides is extremely large and, consequently, the coupling is very weak.

The purpose of our study is to develop a general theory for the coupling between two identical rectangular waveguides. We will present a different

and more general method than that of Miller and Marcatili, that is valid for a strong coupling, i.e. for an arbitrary separation distance. In our calculation, we consider the reflection and transmission coefficients between the two guides and use their exact values in our differential equations for the power transfer between the two guides. We prove that these coefficients are complex quantities with a nonzero real part. Unlike us, Miller defined a coupling coefficient which, in his solution, turns out to be a purely imaginary quantity.

The geometrical description of the system of two guides and the statement of our problem are given in Chapter I.

In Chapter II, we summarize Marcatili's study [1] of the single rectangular waveguide. However, in our presentation we use the concept of ray optics; since the electric and magnetic fields are treated as incident, reflected and transmitted waves, we use exponential forms for the fields inside the waveguide instead of the sinusoidal forms used by Marcatili.

In Chapter III, we study the system of two waveguides. In the coupling region $(0 \leq z \leq L)$, we calculate the reflection and transmission coefficients at the inner boundaries and find the eigenvalue equations for modes. The Miller-Marcatili coupling solution is also discussed in this chapter.

Chapter IV is reserved for a presentation of our method for coupling calculations between the two guides, with a discussion and comparison with the Miller-Marcatili method.

A presentation of the main results of our analysis was given in [4].

CHAPTER I

PROBLEM PRESENTATION AND GEOMETRICAL DESCRIPTION

Rectangular waveguides are made of dielectric material immersed in substrates of different dielectric materials with refractive indexes smaller than that of the guide. If this condition is obeyed, there are angles of incidence at the boundary core substrate for which total internal reflection occurs, as a result of which the light remains contained in the waveguide. Figs. 1-a and 1-b show the system of single rectangular waveguide and two parallel rectangular waveguides, respectively.

For the system of two waveguides, the guides are assumed to be of identical cross section and made of the same material with refractive index n_1, while the refractive index of the substrate is $n_2 < n_1$. Also, the waveguides are assumed to be buried inside the substrate at a sufficiently deep distance, so that we can neglect the effect of the boundary air-substrate. Fig. 2 shows the two guides in the transverse plane (x,y). The two rectangular cross sections are surrounded by the materials of the substrate. We consider the more general case where the materials on all four sides of the guides have different refractive indexes, as shown in Fig. 2.

We define:

Region 1 - where the refractive index $n = n_1$,

Region 2 - where the refractive index $n = n_2$,

Region 3 - where the refractive index $n = n_3$,

Region 4 - where the refractive index $n = n_4$, and

Region 5 - where the refractive index $n = n_5$.

The separation distance between the two guides is c, the width of each guide in the y-direction is b, and the height in the x-direction is a.
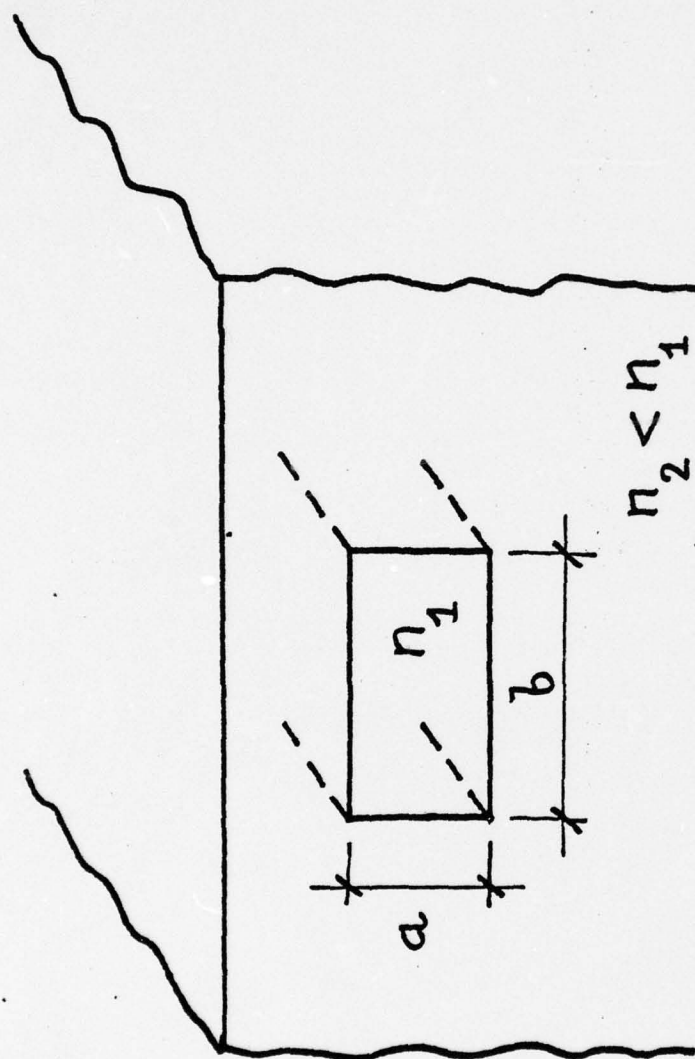
Figure 1a. Isolated rectangular optical waveguide embedded in substrate.
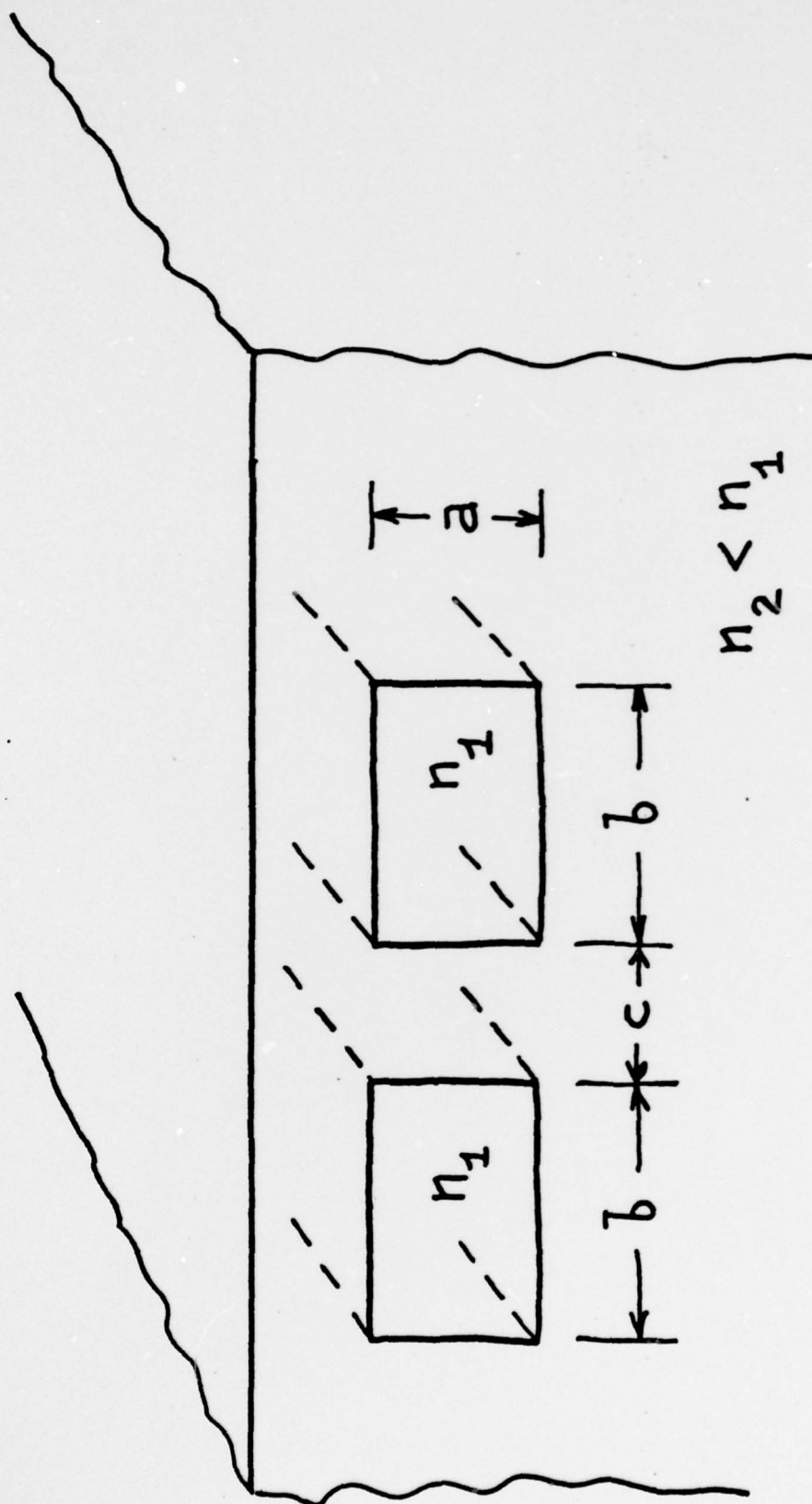
Figure 1b. System of two parallel waveguides.
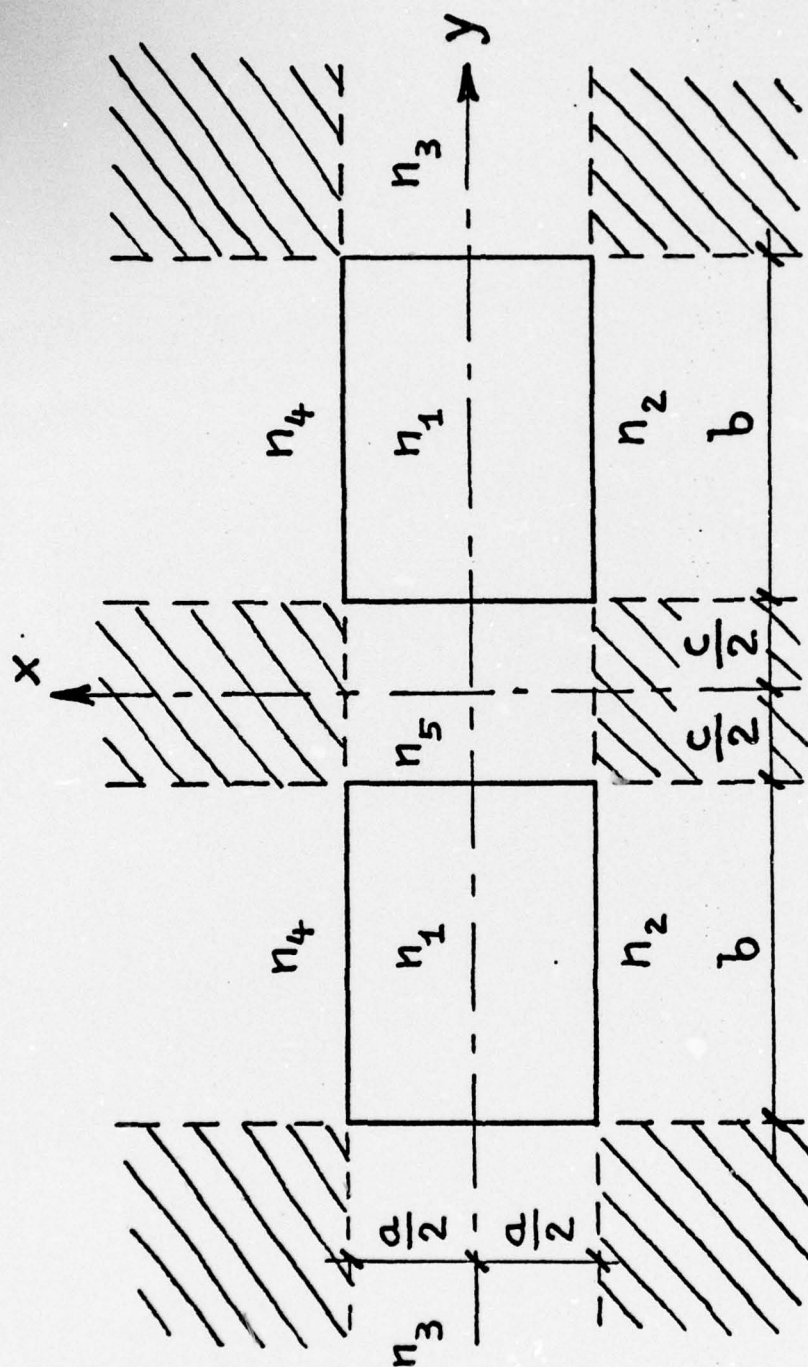
Figure 2. Cross section of coupling region.

Fig. 3 shows the two guides in the y-z plane. In the region ($z \leq 0$) we have only one guide (we call it guide I); in the region ($z \geq L$) we have the other guide (we call it guide II); in the region ($0 \leq z \leq L$) both guides are present (we call it coupling region). In the coupling region, a transfer of power occurs back and forth between the two guides. We call L the coupling distance.

We assume that we have a wave inside guide I traveling in the positive z-direction, and that the condition of total internal reflection is satisfied, so that all the optical power is contained in guide I.

When the wave reaches the boundary $z = 0$ where guide II begins, we have a transfer of power from guide I to guide II. Traveling waves occur in each of the two guides with amplitudes changing along the z-axis. At the boundary $z = L$, the wave in guide I is partially reflected, and the wave in guide II continues to propagate in the positive z-direction. Because of the fact that guide I and guide II are identical, the guided modes in guide II (for $z \geq L$) are the same as the guided modes in guide I (for $z \leq 0$). As a consequence, the wave in guide II (for $z \geq L$) will carry the original incident mode and the condition of total internal reflection will also be satisfied.

Inside guide I, we call incident wave the wave reaching the boundary ($z = 0$) from the left; inside guide II, the wave traveling to the right of the boundary $z = L$ is called the transmitted wave.

We assume that we know the power associated with the incident wave, and we want to determine the transmitted power in function of the separation distance c and of the coupling distance L.

In our study, we consider the case where the power associated with the partially reflected wave at the boundary $z = L$ is very small compared to the transmitted power; hence, we ignore it, which means that we will be dealing only with waves traveling in the positive z-direction.
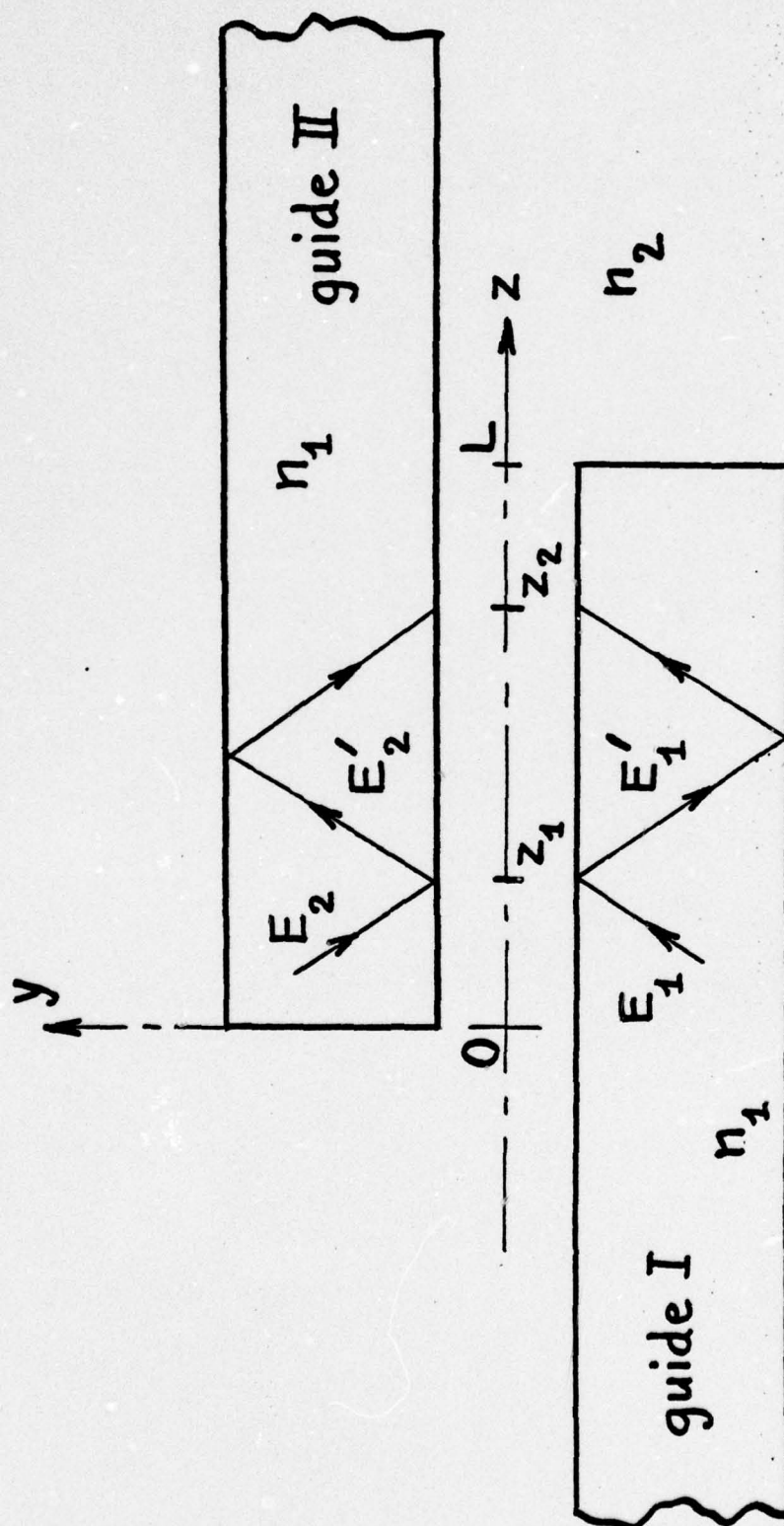
Figure 3. Top view of coupling region.

To determine the electric and magnetic fields and to find the guided modes, we solve Maxwell's equations in three different regions ($z \leq 0$, $0 \leq z \leq L$ and $z \geq L$), and impose the boundary conditions. Then, the transmitted power in function of the incident power is calculated.

In the regions ($z \leq 0$, $z \geq L$) we have to study the single rectangular waveguide. As we mentioned in the introduction, this study was done by Marcatili [1], and will be summarized in Chapter II. The electric and magnetic fields will be written down in exponential form inside the guiding core; thus, the form of our equations will be different from Marcatili's. In the region ($0 \leq z \leq L$), we have two guides, and the solution will be worked out in Chapters III and IV.

CHAPTER II

RECTANGULAR DIELECTRIC WAVEGUIDE

We will study the rectangular waveguide shown in Fig. 3 in the region $(z \leq 0)$. In the x-y plane, the waveguide has the cross section represented by the rectangle at the left of the x-axis of Fig. 2.

We assume that the time and z-dependence are given by the factor:

$$e^{-i(\beta_0 z - \omega t)}. \tag{1}$$

Where ω is the angular frequency, it is given by the relation:

$$k^2 = \omega^2 \varepsilon_0 \mu_0, \tag{2}$$

where $k = \dfrac{2\pi}{\lambda}$ is the wave vector in free space. The quantity β_0 is the propagation constant along the z-axis in the region $(z \leq 0)$. The factor of eq. (1) will be omitted in the following.

We will consider the case where:

$$\beta_0 \simeq n_1 k. \tag{3}$$

This corresponds to modes far from cutoff; thus, almost all the power is contained inside the core of the waveguide. In this case we can neglect the fields in the shaded areas of Fig. 2.

2.1 Guided Modes of the Rectangular Waveguide.

To get a complete discription of the guided modes inside the waveguide, we must solve Maxwell's equations:

$$\nabla \times \vec{H} = \varepsilon_0 n^2 \frac{\delta \vec{E}}{\delta t}. \tag{4}$$

$$\nabla \times \vec{E} = -\mu_0 \frac{\delta \vec{H}}{\delta t}. \tag{5}$$

where the operator ∇ is given by:

$$\nabla = \frac{\delta}{\delta x}\,\hat{e}_x + \frac{\delta}{\delta y}\,\hat{e}_y + \frac{\delta}{\delta z}\,\hat{e}_z, \tag{6}$$

and n is the refractive index.

Considering relation (1), we can write the transverse field components in function of the longitudinal components as:

$$E_x = -\frac{i}{K_j^2}\left[\beta_0\frac{\partial E_z}{\partial x} + \omega\mu_0\frac{\partial H_z}{\partial y}\right] \tag{7}$$

$$E_y = -\frac{i}{K_j^2}\left[\beta_0\frac{\partial E_z}{\partial y} - \omega\mu_0\frac{\partial H_z}{\partial x}\right] \tag{8}$$

$$H_x = -\frac{i}{K_j^2}\left[\beta_0\frac{\partial H_z}{\partial x} - \omega n_j^2\epsilon_0\frac{\partial E_z}{\partial x}\right] \tag{9}$$

$$H_y = -\frac{i}{K_j^2}\left[\beta_0\frac{\partial H_z}{\partial y} + \omega n_j^2\epsilon_0\frac{\partial E_z}{\partial y}\right] \tag{10}$$

Here j indicates the region number in the x-y plane, and:

$$K_j^2 = n_j^2 k^2 - \beta_0^2. \tag{11}$$

The longitudinal components E_z and H_z must satisfy the reduced wave equation:

$$\frac{\delta^2\psi}{\delta x^2} + \frac{\delta^2\psi}{\delta y^2} + K_j^2\psi = 0. \tag{12}$$

We can always choose the amplitude of the longitudinal components so that one of the transverse components is zero. The choice ($H_x = 0$) corresponds to modes with polarization predominantly in the x-direction. These modes are called E^x_{pq} modes. The choice ($H_y = 0$) leads to modes with polarization predominantly in the y-direction. These modes are called E^y_{pq} modes.

In this paper, we consider only E^x_{pq} modes, but the same analysis can obviously be applied for E^y_{pq} modes.

2.1.1 Waveguide in Region ($z \leq 0$).

Inside the core we can write the electric and magnetic fields with real propagation constants in the x- and y-directions. For the two logitudinal components E_z and H_z we write:

$$E_z = A[e^{-iK_x x} + A' e^{iK_x x}][e^{-iK_y y} + A'' e^{iK_y y}] \tag{13}$$

$$H_z = A\sqrt{\frac{\epsilon_0}{\mu_0}} n_1^2 \frac{K_y}{K_x} \cdot \frac{k}{\beta_0}[e^{-iK_x x} - A' e^{iK_x x}][e^{-iK_y y} - A'' e^{iK_y y}] \tag{14}$$

where K_x and K_y are the propagation constants in x- and y-directions respectively.

This set of fields corresponds to $H_x = 0$. For the reduced wave equation (12) to be satisfied, the propagation constants must obey the relation:

$$K_1^2 = n_1^2 k^2 - \beta_0^2 = K_x^2 + K_y^2 \tag{15}$$

Substituting in (7) to (10), we find the transverse field components:

$$E_x = -\frac{A}{K_x \beta_0}[n_1^2 k^2 - K_x^2][e^{-iK_x x} - A' e^{iK_x x}][e^{-iK_y y} + A'' e^{iK_y y}] \tag{16}$$

$$E_y = -A\frac{K_y}{\beta_0}[e^{-iK_x x} + A' e^{iK_x x}][e^{-iK_y y} - A'' e^{iK_y y}] \tag{17}$$

$$H_x \equiv 0 \tag{18}$$

$$H_y = -A\sqrt{\frac{\epsilon_0}{\mu_0}} n_1^2 \frac{k}{K_x}[e^{-iK_x x} - A' e^{iK_x x}][e^{-iK_y y} + A'' e^{iK_y y}] \tag{19}$$

Considering relation (3), we find that K_x or $K_y \ll \beta_0$. If we compare the electric field components we find:

$$E_x \sim \frac{\beta}{K_x} E_z \sim \frac{\beta^2}{K_x K_y} E_y. \tag{20}$$

The most significant field component is E_x, which means that the wave is polarized predominantly in the x-direction; E_y is the least significant component and it will be ignored.

Outside the core the fields must vanish at a large distance from the waveguide. The fields must be represented by an exponentially decaying function, which means that the propagation constant must be purely imaginary.

We assume that the x-dependence in regions 3 and 5 is the same as in region 1, and that the y-dependence in regions 2 and 4 is the same as in region 1.

With the requirement that the tengential electric field components are continuous across the core boundaries, the field components outside the core can be written as follows:

Region 2:

$$E_z = A[e^{iK_x\frac{a}{2}} + A' e^{-iK_x\frac{a}{2}}] e^{\gamma_2(x+\frac{a}{2})} [e^{-iK_y y} + A'' e^{iK_y y}] \tag{21}$$

$$H_z = -iA\sqrt{\frac{\epsilon_0}{\mu_0}}\, n_2^2 \frac{K_y}{\gamma_2} \cdot \frac{k}{\beta_0} [e^{iK_x\frac{a}{2}} + A' e^{-iK_x\frac{a}{2}}] e^{\gamma_2(x+\frac{a}{2})} [e^{-iK_y y} - A'' e^{iK_y y}] \tag{22}$$

$$E_x = -iA \frac{\gamma_2^2 + n_2^2 k^2}{\gamma_2 \beta_0} [e^{iK_x\frac{a}{2}} + A' e^{-iK_x\frac{a}{2}}] e^{\gamma_2(x+\frac{a}{2})} [e^{-iK_y y} + A'' e^{iK_y y}] \tag{23}$$

$$E_y \simeq 0 \quad , \quad H_x \cong 0$$

$$H_y = iA\sqrt{\frac{\epsilon_0}{\mu_0}}\, n_2^2 \frac{k}{\gamma_2} [e^{iK_x\frac{a}{2}} + A' e^{-iK_x\frac{a}{2}}] e^{\gamma_2(x+\frac{a}{2})} [e^{-iK_y y} + A'' e^{iK_y y}] \tag{24}$$

with

$$K_2^2 = n_2^2 k^2 - \beta_0^2 = K_y^2 - \gamma_2^2 \tag{25}$$

Region 4:

$$E_z = A\left[e^{-iK_x\frac{a}{2}} + A'e^{iK_x\frac{a}{2}}\right]e^{-\gamma_4(x-\frac{a}{2})}\left[e^{-iK_yy} + A''e^{iK_yy}\right] \tag{26}$$

$$H_z = iA\sqrt{\frac{\epsilon_0}{\mu_0}}\,n_4^2\,\frac{K_y}{\gamma_4}\cdot\frac{k}{\beta_0}\left[e^{-iK_x\frac{a}{2}} + A'e^{iK_x\frac{a}{2}}\right]e^{-\gamma_4(x-\frac{a}{2})}\left[e^{-iK_yy} - A''e^{iK_yy}\right] \tag{27}$$

$$E_x = \frac{iA}{\gamma_4\beta_0}\left[n_4^2 k^2 + \gamma_4\right]\left[e^{-iK_x\frac{a}{2}} + A'e^{iK_x\frac{a}{2}}\right]e^{-\gamma_4(x-\frac{a}{2})}\left[e^{-iK_yy} + A''e^{iK_yy}\right] \tag{28}$$

$$E_y \simeq 0 \quad , \quad H_x \equiv 0$$

$$H_y = -iA\sqrt{\frac{\epsilon_0}{\mu_0}}\,n_4^2\,\frac{k}{\gamma_4}\left[e^{-iK_x\frac{a}{2}} + A'e^{iK_x\frac{a}{2}}\right]e^{-\gamma_4(x-\frac{a}{2})}\left[e^{-iK_yy} + A''e^{iK_yy}\right] \tag{29}$$

with

$$K_4^2 = n_4^2 k^2 - \beta_0^2 = K_y^2 - \gamma_4^2 \tag{30}$$

In regions 3 and 5, we adjust the amplitudes so that the strong component E_x is continuous at the core boundaries. E_z is a second order quantity; thus we obtain:

Region 3:

$$E_z = A\frac{n_1^2}{n_3^2}\left[e^{iK_y(b+\frac{c}{2})} + A''e^{-iK_y(b+\frac{c}{2})}\right]\left[e^{-iK_xx} + A'e^{iK_xx}\right]e^{\gamma_3(y+b+\frac{c}{2})} \tag{31}$$

$$H_z = iA\sqrt{\frac{\epsilon_0}{\mu_0}}\,n_1^2\frac{k}{\beta_0}\cdot\frac{\gamma_3}{K_x}\left[e^{iK_y(b+\frac{c}{2})} + A''e^{-iK_y(b+\frac{c}{2})}\right]\left[e^{-iK_xx} - A'e^{iK_xx}\right]e^{\gamma_3(y+b+\frac{c}{2})} \tag{32}$$

$$E_x = -A\frac{n_1^2}{n_3^2}\left[\frac{n_3^2 k^2 - K_x^2}{\beta_0 K_x}\right]\left[e^{iK_y(b+\frac{c}{2})} + A''e^{-iK_y(b+\frac{c}{2})}\right]\left[e^{-iK_xx} - A'e^{iK_xx}\right]e^{\gamma_3(y+b+\frac{c}{2})} \tag{33}$$

$$H_x \equiv 0 \quad , \quad E_y \simeq 0$$

$$H_y = -A\sqrt{\tfrac{\epsilon_0}{\mu_0}}\, n_1^2 \frac{k}{K_x}\left[e^{iK_y(b+\frac{c}{2})} + A'' e^{-iK_y(b+\frac{c}{2})}\right]\left[e^{-iK_x X} - A' e^{iK_x X}\right] e^{\gamma_3(y+b+\frac{c}{2})} \tag{34}$$

with

$$K_3^2 = n_3^2 k^2 - \beta_0^2 = K_x^2 - \gamma_3^2 \tag{35}$$

Region 5:

$$E_z = A\frac{n_1^2}{n_5^2}\left[e^{iK_y\frac{c}{2}} + A'' e^{-iK_y\frac{c}{2}}\right]\left[e^{-iK_x X} + A' e^{iK_x X}\right] e^{-\gamma_5(y+\frac{c}{2})} \tag{36}$$

$$H_z = -iA\sqrt{\tfrac{\epsilon_0}{\mu_0}}\, n_1^2 \frac{k}{\beta_0}\cdot\frac{\gamma_5}{K_x}\left[e^{iK_y\frac{c}{2}} + A'' e^{-iK_y\frac{c}{2}}\right]\left[e^{-iK_x X} - A' e^{iK_x X}\right] e^{-\gamma_5(y+\frac{c}{2})} \tag{37}$$

$$E_x = -A\frac{n_1^2}{n_5^2}\left[\frac{n_5^2 k^2 - K_x^2}{\beta_0 K_x}\right]\left[e^{iK_y\frac{c}{2}} + A'' e^{-iK_y\frac{c}{2}}\right]\left[e^{-iK_x X} - A' e^{iK_x X}\right] e^{-\gamma_5(y+\frac{c}{2})} \tag{38}$$

$$E_y \simeq 0 \quad , \quad H_x \equiv 0$$

$$H_y = -A\sqrt{\tfrac{\epsilon_0}{\mu_0}}\, n_1^2 \frac{k}{K_x}\left[e^{iK_y\frac{c}{2}} + A'' e^{-iK_y\frac{c}{2}}\right]\left[e^{-iK_x X} - A' e^{iK_x X}\right] e^{-\gamma_5(y+\frac{c}{2})} \tag{39}$$

with

$$K_5^2 = n_5^2 k^2 - \beta_0^2 = K_x^2 - \gamma_5^2 \tag{40}$$

We note here that the E_x component is continuous provided that we neglect K_x^2 compared to $n_j^2 k^2$ in the numerator of equations (33) and (38).

We know that the tengential field components must be matched at each boundary separating the different regions.

In the x-direction, at the boundaries $x = -\frac{a}{2}$ and $x = \frac{a}{2}$, the tangential field components are E_z, H_z, E_y and H_y. E_z is already matched. E_y is negligible and by matching H_z, H_y automatically will be matched.

In the y-direction, at the boundaries $y = -b - \frac{c}{2}$ and $y = -\frac{c}{2}$, the tangential field components are E_z, H_z, E_x and H_x. E_x is already continuous, H_x is zero, E_z is a second order quantity and we will neglect it here. H_z is the only remaining tangential component to be matched.

This leads to the following:

at $x = -\frac{a}{2}$:

$$\frac{n_1^2}{K_x}\left[e^{iK_x\frac{a}{2}} - A'e^{-iK_x\frac{a}{2}}\right] = -i\,\frac{n_2^2}{\gamma_2}\left[e^{iK_x\frac{a}{2}} + A'e^{-iK_x\frac{a}{2}}\right] \tag{41}$$

at $x = \frac{a}{2}$:

$$\frac{n_1^2}{K_x}\left[e^{-iK_x\frac{a}{2}} - A'e^{iK_x\frac{a}{2}}\right] = i\,\frac{n_4^2}{\gamma_4}\left[e^{-iK_x\frac{a}{2}} + A'e^{iK_x\frac{a}{2}}\right] \tag{42}$$

Considering $e^{iK_x\frac{a}{2}}$ and $e^{-iK_x\frac{a}{2}}$ as unknowns, eqs. (41) and (42) can be rewritten as follows:

$$\left[\frac{n_1^2}{K_x} + \frac{in_2^2}{\gamma_2}\right]e^{iK_x\frac{a}{2}} - A'\left[\frac{n_1^2}{K_x} - i\,\frac{n_2^2}{\gamma_2}\right]e^{-iK_x\frac{a}{2}} = 0 \tag{43}$$

$$A'\left[\frac{n_1^2}{K_x} + i\,\frac{n_4^2}{\gamma_4}\right]e^{iK_x\frac{a}{2}} - \left[\frac{n_1^2}{K_x} - i\,\frac{n_4^2}{\gamma_4}\right]e^{-iK_x\frac{a}{2}} = 0 \tag{44}$$

To have a solution different from zero, the determinant of the coefficients must be zero. This leads to:

$$A'^2 = \left[\frac{n_1^2}{K_x} - i\,\frac{n_4^2}{\gamma_4}\right]\left[\frac{n_1^2}{K_x} + i\,\frac{n_2^2}{\gamma_2}\right] \Bigg/ \left[\frac{n_1^2}{K_x} - i\,\frac{n_2^2}{\gamma_2}\right]\left[\frac{n_1^2}{K_x} + i\,\frac{n_4^2}{\gamma_4}\right] \tag{45}$$

Substituting (45) in (41) or (42) and solving for $\tan K_x$ we find:

$$\tan K_x a = \frac{n_1^2 K_x\left[n_4^2 \gamma_2 + n_2^2 \gamma_4\right]}{\left[n_4^2 n_2^2 K_x^2 - n_1^4 \gamma_2 \gamma_4\right]} \tag{46}$$

This is the eigenvalue equation of the TM modes of the infinite slab waveguide. In the particular case when $n_4 = n_2$ we have $\gamma_4 = \gamma_2$, and substituting in (45) we find:

$$A'^2 = 1 \text{ or } A' = \pm 1. \tag{47}$$

For $A' = -1$, we substitute in (41) and we find the even modes equation:

$$\tan K_x \frac{a}{2} = \frac{n_1^2}{n_2^2} \cdot \frac{\gamma_2}{K_x}. \tag{48}$$

For $A' = +1$, we have the odd modes equation:

$$\tan K_x \frac{a}{2} = \frac{n_2^2}{n_1^2} \cdot \frac{K_x}{\gamma_2}. \tag{49}$$

In matching H_z in the y-direction we get:

at $y = -(b + \frac{c}{2})$:

$$K_y\left[e^{iK_y(b+\frac{c}{2})} - A''e^{-iK_y(b+\frac{c}{2})}\right] = i\gamma_3\left[e^{iK_y(b+\frac{c}{2})} + A''e^{-iK_y(b+\frac{c}{2})}\right] \tag{50}$$

at $y = -\frac{c}{2}$:

$$K_y\left[e^{iK_y\frac{c}{2}} - A''e^{-iK_y\frac{c}{2}}\right] = -i\gamma_5\left[e^{iK_y\frac{c}{2}} + A''e^{-iK_y\frac{c}{2}}\right] \tag{51}$$

From (50) we find:

$$e^{iK_y(c+2b)} = A'' \frac{K_y - i\gamma_3}{K_y + i\gamma_3} = A'' e^{-2i \arctan \frac{\gamma_3}{K_y}} \tag{52}$$

From (51) we find:

$$e^{iK_y c} = A'' \frac{K_y - i\gamma_5}{K_y + i\gamma_5} = A'' e^{-2i \arctan \frac{\gamma_5}{K_y}} \tag{53}$$

Assuming $A'' \neq 0$, we find:

$$\frac{(52)}{(53)} = e^{2iK_y b} = e^{2i\left(\arctan \frac{\gamma_3}{K_y} + \arctan \frac{\gamma_5}{K_y}\right)} \tag{54}$$

or

$$2iK_y b = 2i \arctan \frac{\gamma_3}{K_y} + 2i \arctan \frac{\gamma_5}{K_y} + 2in\pi \tag{55}$$

which is the eigenvalue equation of the TE modes of the infinite slab wave-guide. We can write it as:

$$\tan K_y b = \frac{K_y(\gamma_3 + \gamma_5)}{K_y^2 - \gamma_3\gamma_5}. \tag{56}$$

For the particular case when $n_3 = n_5$ we have $\gamma_3 = \gamma_5$ and (56) becomes:

$$\tan K_y b = \frac{2K_y\gamma_5}{K_y^2 - \gamma_5^2}. \tag{57}$$

or

$$\tan K_y \frac{b}{2} = \frac{\gamma_5}{K_y}, \tag{58}$$

$$\tan K_y \frac{b}{2} = -\frac{K_y}{\gamma_5}; \tag{59}$$

(58) and (59) are the eigenvalue equations for the even and odd TE modes of the infinite slab waveguide, respectively.

From now on in our analysis we will assume that $n_2 = n_3 = n_4 = n_5$, which is the case of a rectangular waveguide immersed in a homogeneous and isotropic substrate.

The constant A'' can be determined as follows.

Let us first consider the even modes in the y-direction. We can write (58) as:

$$e^{-2i \arctan \frac{\gamma_5}{K_y}} = e^{-i \, K_y b};$$
(60)

substituting in (53) we find:

$$A'' = e^{iK_y(c+b)}.$$
(61)

For the odd modes we have:

$$e^{2i \arctan \frac{K_y}{\gamma_5}} = e^{-i \, K_y b};$$
(62)

also, we can write (53) as:

$$e^{i \, K_y c} = -A'' \, e^{2i \arctan \frac{K_y}{\gamma_5}};$$
(63)

substituting (62) in (63) we find:

$$A'' = -e^{iK_y(b+c)}.$$
(64)

Finally, we see that the constants A' and A'' can be determined from (47), (61) and (64) depending on the kind of modes. The only remaining unknown constant in the expressions of field components is A, which will be determined from the amount of power traveling in the waveguide.

The eigenvalues for the propagation constant K_x determined by (46) and the eigenvalues for K_y determined by (57) remain the same along the z-axis as long as there is no change in the physical parameters of the waveguide.

or graphically. Using the second way, the solution for the eigenvalue equation (57) is presented in Fig. 4. As a variable we use $x = K_y b$, and the solution is determined by the intersection of the branch $\tan x$ with $\dfrac{2K_y \gamma_5}{K_y^2 - \gamma_5^2}$; $\gamma_5(x)$ can be obtained by subtracting eq. (25) from eq. (15):

$$(n_1^2 - n_2^2)k^2 = K_y^2 + \gamma_5^2, \tag{65}$$

or:

$$\gamma_5(x) = \sqrt{(n_1^2 - n_2^2)k^2 - K_y^2} = \frac{1}{b}\sqrt{(n_1^2 - n_2^2)k^2 b^2 - K_y^2 b^2}$$

$$= \frac{1}{b}\sqrt{(n_1^2 - n_2^2)k^2 b^2 - x^2} \tag{66}$$

and:

$$\frac{\gamma_5}{K_y} = \frac{\sqrt{(n_1^2 - n_2^2)k^2 b^2 - x^2}}{x} \tag{67}$$

This solution determines the propagation constants in regions $(z \leq 0)$ and $(z \geq L)$. Some of the curves in Fig. 4 were drawn for $(n_1^2 - n_2^2)k^2 b^2 = 100$. Under this condition the modes are cut off for:

$$\sqrt{(n_1^2 - n_2^2)k^2 b^2 - x^2} = \sqrt{100 - x^2} \leq 0 \tag{68}$$

or:

$$x \geq 10. \tag{69}$$

We see that there are eight intersections, and this guide can carry eight modes.

For $(n_1^2 - n_2^2)k^2 b^2 = 10$, the curves drawn in Fig. 4 show that only four modes can be carried by the guide. By decreasing $(n_1^2 - n_2^2)k^2 b^2$, we decrease the number of possible modes in the guide.
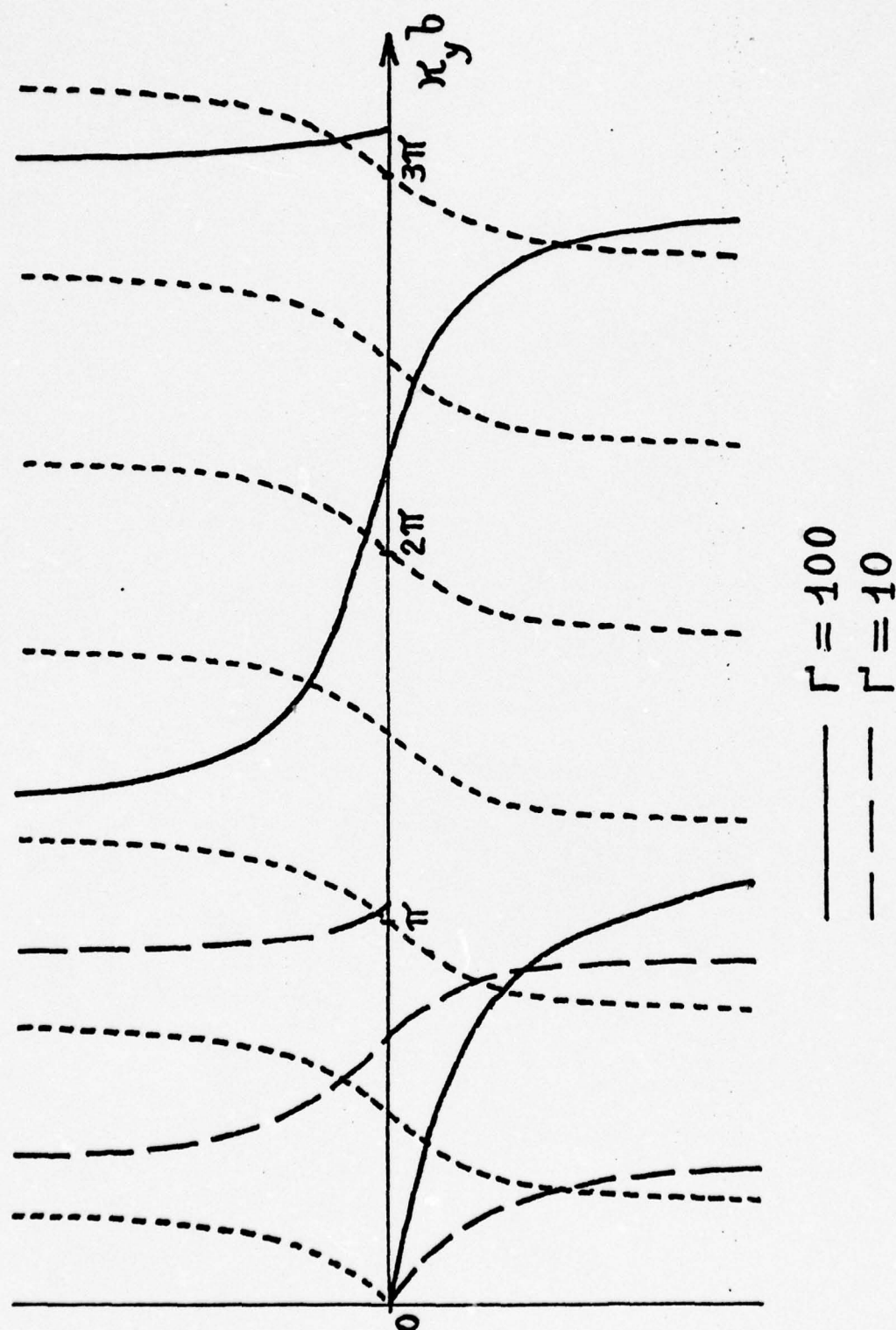
Figure 4. Eigenvalues for single rectangular waveguide, for different values of $\Gamma = (n_1^2 - n_2^2)k^2b^2$.

When we reach the region ($z \gtrsim 0$) (see Fig. 3), we have both guide I and guide II present. The boundary conditions in the x-direction are the same as in region ($z \leq 0$); as a consequence, the modes in the x-direction and the eigenvalues for K_x will remain the same as in region ($z \leq 0$). In the y-direction we will have an exchange of power between the two waveguides through the separation region, by means of evanescent waves. The boundary conditions which we must impose on the fields in this region are different from the conditions we imposed in region ($z \leq 0$); as a consequence, the eigenvalue equations (58) and (59) are not valid in this region, and the propagation constant K_y will be determined by new eigenvalue equations. The new K_y will be a function of the separation distance c. We note here that the propagation constants in region ($z \geq L$) are the same as in region ($z \leq 0$).

The above analysis shows that light will propagate in region ($0 \leq z \leq L$) with a propagation constant different from that of regions ($z \leq 0$) and ($z \leq L$). This means that we will have a reflection in guide I at the boundary ($z = 0$), and a reflection in guide II at the boundary ($z = L$). Another effect can take place if the guides were multimode carriers. This effect is the excitation of undesired modes and coupling between different modes.

For maximum power transfer from guide I to guide II, the two reflections at ($z = 0$) and at ($z = L$) must be minimized, which means that we have to look for optimal conditions so that the difference between the propagation constants in the different regions along z is minimum. Also, the excitation of undesired modes can be eliminated if the propagation constants of the desired modes are the same in all three regions.

In the light of the above discussion, we need to find the propagating modes in the three different regions. This is done by solving the corresponding eigenvalue equations. The solution can be obtained either numerically

In the region $(0 \leq z \leq L)$, the solution will be presented in Chapter III and Fig. 5.

2.1.2 Waveguide in Region $(z \geq L)$.

For guide II in region $(z \geq L)$ and for both guides in region $(0 \leq z \leq L)$, the boundary conditions in the x-direction are the same as for guide I in region $(0 \geq z)$. As a consequence, the x-dependence of the field components, the eigenvalue equations for modes and the propagation constants in the x-direction are the same in all three regions. From now on we drop the x-dependence of the field components in our equations.

However, in the y-direction the boundaries are not the same in all three regions, and therefore the y-dependence, the eigenvalue equations for the modes and the propagation constants also vary from region to region.

For the case of guide II in region $(z \geq L)$, we can write the field components as:

Region 1:

$$E_z = B[e^{-iK_y y} + B'' e^{iK_y y}] \tag{70}$$

$$H_z = B\sqrt{\frac{\epsilon_0}{\mu_0}}\, n_i^2 \frac{K_y}{K_x} \cdot \frac{k}{\beta_0} [e^{-iK_y y} - B'' e^{iK_y y}] \tag{71}$$

$$E_x = -\frac{B}{K_x \beta_0}[n_i^2 k^2 - K_x^2][e^{-iK_y y} + B'' e^{iK_y y}] \tag{72}$$

$$E_y \simeq 0 \quad , \quad H_x \equiv 0$$

$$H_y = -B\sqrt{\frac{\epsilon_0}{\mu_0}} n_i^2 \frac{k}{K_x}[e^{-iK_y y} + B'' e^{iK_y y}] \tag{73}$$

Region 3:

$$E_z = B \frac{n_i^2}{n_3^2} \left[e^{-iK_y(b+\frac{c}{2})} + B'' e^{iK_y(b+\frac{c}{2})} \right] e^{-\gamma_3(y-b-\frac{c}{2})} \tag{74}$$

$$H_z = -iB\sqrt{\frac{\epsilon_o}{\mu_o}} n_i^2 \frac{k}{\beta_o} \cdot \frac{\gamma_3}{K_x} \left[e^{-iK_y(b+\frac{c}{2})} + B'' e^{iK_y(b+\frac{c}{2})} \right] e^{-\gamma_3(y-b-\frac{c}{2})} \tag{75}$$

$$E_x = -B \frac{n_i^2}{n_3^2} \frac{n_3^2 k^2 - K_x^2}{\beta_o K_x} \left[e^{-iK_y(b+\frac{c}{2})} + B'' e^{iK_y(b+\frac{c}{2})} \right] e^{-\gamma_3(y-b-\frac{c}{2})} \tag{76}$$

$$E_y \simeq 0 \quad , \quad H_x \equiv 0$$

$$H_y = -B\sqrt{\frac{\epsilon_o}{\mu_o}} n_i^2 \frac{k}{K_x} \left[e^{-iK_y(b+\frac{c}{2})} + B'' e^{iK_y(b+\frac{c}{2})} \right] e^{-\gamma_3(y-b-\frac{c}{2})} \tag{77}$$

Region 5:

$$E_z = B \frac{n_i^2}{n_5^2} \left[e^{-iK_y\frac{c}{2}} + B'' e^{iK_y\frac{c}{2}} \right] e^{\gamma_5(y-\frac{c}{2})} \tag{78}$$

$$H_z = iB\sqrt{\frac{\epsilon_o}{\mu_o}} n_i^2 \frac{k}{\beta_o} \cdot \frac{\gamma_5}{K_x} \left[e^{-iK_y\frac{c}{2}} + B'' e^{iK_y\frac{c}{2}} \right] e^{\gamma_5(y-\frac{c}{2})} \tag{79}$$

$$E_x = -B \frac{n_i^2}{n_5^2} \cdot \frac{n_5^2 k^2 - K_x^2}{\beta_o K_x} \left[e^{-iK_y\frac{c}{2}} + B'' e^{iK_y\frac{c}{2}} \right] e^{\gamma_5(y-\frac{c}{2})} \tag{80}$$

$$E_y \simeq 0 \quad , \quad H_x \equiv 0$$

$$H_y = -B\sqrt{\frac{\epsilon_o}{\mu_o}} n_i^2 \frac{k}{K_x} \left[e^{-iK_y\frac{c}{2}} + B'' e^{iK_y\frac{c}{2}} \right] e^{\gamma_5(y-\frac{c}{2})} \tag{81}$$

H_z is the only component which has to be matched at $y = \frac{c}{2}$ and at $y = b + \frac{c}{2}$. At $y = b + \frac{c}{2}$ we find:

$$K_y\left[e^{-iK_y(b+\frac{c}{2})} - B''e^{iK_y(b+\frac{c}{2})}\right] = -i\gamma_3\left[e^{-iK_y(b+\frac{c}{2})} + B''e^{iK_y(b+\frac{c}{2})}\right] \tag{82}$$

or:

$$e^{-iK_y(2b+c)} = B''e^{-2i\arctan\frac{\gamma_3}{K_y}} \tag{83}$$

At $y = \frac{c}{2}$ we find:

$$K_y\left[e^{-iK_y\frac{c}{2}} - B''e^{iK_y\frac{c}{2}}\right] = i\gamma_5\left[e^{-iK_y\frac{c}{2}} + B''e^{iK_y\frac{c}{2}}\right] \tag{84}$$

or:

$$e^{iK_yc} = B''e^{2i\arctan\frac{\gamma_5}{K_y}} \tag{85}$$

Assuming $B'' \neq 0$ and dividing eq. (85) by eq. (83):

$$e^{2iK_yb} = e^{2i\arctan\frac{\gamma_5}{K_y} + 2i\arctan\frac{\gamma_3}{K_y}} = e^{4i\arctan\frac{\gamma_5}{K_y}} \tag{86}$$

or:

$$\tan K_yb = \frac{2K_y\gamma_5}{K_y^2 - \gamma_5^2}, \text{ and} \tag{87}$$

$$\tan\frac{K_yb}{2} = \frac{\gamma_5}{K_y} \quad \text{or} \quad \tan\frac{K_yb}{2} = -\frac{K_y}{\gamma_5} \tag{88}$$

Eq. (87) is the same as eq. (57), which means that in the y-direction the modes of guide II in region ($z \geq L$) are the same as the modes of guide I in region ($0 \geq z$).

The constant B'' can be obtained from (85); it is given by:

$$B'' = e^{-iK_y(b+c)} \tag{89}$$

for even modes, and by:

$$B^{\prime\prime} = -e^{-iK_y(b+c)} \tag{90}$$

for odd modes.

2.2 Power in Rectangular Waveguide

If we are far from cut-off, the fields outside the core are very small compared to the field inside the core, and almost all the power is contained inside the core. As an approximation, we will assume that all the power is contained inside the core.

The power can be obtained by integrating the z-component of the power flow vector (Poynting's vector)

$$S_z = \frac{1}{2} \, \text{Re}(\vec{E} \times \vec{H}) \cdot \hat{e}_z \tag{91}$$

over the transverse section of the waveguide core. Eq. (91) implies that S_z is a time-averaged quantity.

In our analysis, we considered no reflected wave in the z-direction, and $\beta_0 > 0$. If P_I is the power in guide I, we thus obtain:

$$P_I = \frac{1}{2} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{b}{2}-\frac{c}{2}}^{-\frac{c}{2}} dy \, \text{Re}\left[E_x H_y^* - E_y H_x^*\right] , \tag{92}$$

but we know that $H_x \equiv 0$ and $E_y \simeq 0$, and with the help of (16) and (19) we find:

$$P_I = \frac{1}{2} \, \text{Re} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-b-\frac{c}{2}}^{-\frac{c}{2}} dy \, (AA^*) \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{k n_i^2}{\beta_0 K_x^2} \left[n_i^2 k^2 - K_x^2\right] \left[e^{-iK_x x} - A' e^{iK_x x}\right] \times$$

$$\left[e^{iK_x x} - A'^* e^{-iK_x x}\right] \left[e^{-iK_y y} + A'' e^{iK_y y}\right] \left[e^{iK_y y} + A''^* e^{-iK_y y}\right] \tag{93}$$

$$P_I = -\frac{1}{2}|A|^2\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{n_i^2 k}{\beta_0 K_x^2}\left[n_i^2 k^2 - K_x^2\right]\int_{-\frac{a}{2}}^{\frac{a}{2}}\left[-1-|A'|^2 + A'e^{2iK_x X} + A'^* e^{-2iK_x X}\right]dX$$

$$\times \int_{-b-\frac{c}{2}}^{-\frac{c}{2}}\left[1+|A''|^2 + A''e^{2iK_y y} + A''^* e^{-2iK_y y}\right]dy \tag{94}$$

We consider relations (47), (61) and (64), the constant $A' = -1$ or $+1$ corresponding to even or odd modes respectively, and $A'' = e^{iK_y(b+c)}$ or $-e^{-iK_y(b+c)}$ corresponding to even or odd modes respectively, and we substitute A' and A'' in (94). We find that

$$P_I = -\frac{1}{2}|A|^2\sqrt{\frac{\epsilon_0}{\mu_0}}\cdot\frac{n_i^2 k}{\beta_0 K_x^2}\left[n_i^2 k^2 - K_x^2\right]\int_{-\frac{a}{2}}^{\frac{a}{2}}\left[-2 \mp \cos 2K_x X\right]dX$$

$$\times \int_{-b-\frac{c}{2}}^{-\frac{c}{2}}\left[2 \pm \cos 2K_y\left(y+\frac{c+b}{2}\right)\right]dy \tag{95}$$

After performing the integrations we find:

$$P_I = 2|A|^2\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{n_i^2 k}{\beta_0 K_x^2}\left[n_i^2 k^2 - K_x^2\right]\left[a \pm \frac{\sin K_x a}{K_x}\right]\left[b \pm \frac{\sin K_y b}{K_y}\right] \tag{96}$$

or:

$$P_I = 2|A|^2\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{n_i^2 k}{\beta_0 K_x^2}\left[n_i^2 k^2 - K_x^2\right]\left[a \pm \frac{1}{K_x}\cdot\frac{2\tan\frac{K_x a}{2}}{1+\tan^2\frac{K_x a}{2}}\right]\left[b \pm \frac{1}{K_y}\cdot\frac{2\tan\frac{K_y b}{2}}{1+\tan^2\frac{K_y b}{2}}\right] \tag{97}$$

The upper sign corresponds to even modes and the lower sign corresponds to odd modes. Substituting for $\tan\frac{K_x a}{2}$ and $\tan\frac{K_y b}{2}$ their respective values

from (48) or (49) and (58) or (59), the right-hand side of (71) is unchanged, and we finally have that:

$$P_I = 2|A|^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{n_1^2 k}{\beta_0 K_x^2} \left[n_1^2 k^2 - K_x^2 \right] \left[a + \frac{2 n_1^2 n_2^2 \gamma_2}{n_2^4 K_x^2 + n_1^4 \gamma_2^2} \right] \left[b + \frac{2 \gamma_5}{K_y^2 + \gamma_5^2} \right] \tag{98}$$

By solving Maxwell's equations and imposing the boundary conditions, we were able to describe the electric and magnetic fields in all regions, and to determine the values of all constants and parameters except for the constant A. The magnitude of this constant can be determined from eq. (98) when we know the amount of power traveling in the waveguide. We see that if the propagation constants remain constant along the waveguide, the power is proportional to the square of the absolute value of A.

In the region ($z \geq L$), the constant B is related to the power P_{II} inside guide II by:

$$P_{II} = 2|B|^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{n_1^2 k}{\beta_0 K_x^2} \left[n_1^2 k^2 - K_x^2 \right] \left[a + \frac{2 n_1^2 n_2^2 \gamma_2}{n_2^4 K_x^2 + n_1^4 \gamma_4^2} \right] \left[b + \frac{2 \gamma_5}{K_y^2 + \gamma_5^2} \right] \tag{99}$$

2.3 Reflection Coefficients.

An important parameter which will be used frequently in the following is the reflection coefficient at the boundary ($y = -\frac{c}{2}$) for guide I and at the boundary ($y = \frac{c}{2}$) for guide II.

To compute this parameter in the case of guide I in region ($z \leq 0$), we see that the y-dependence of the electric field (see relations (13) and (16)) is given by:

$$e^{-iK_y y} + A'' e^{iK_y y} = e^{-iK_y y} \pm e^{iK_y(b+c)} e^{iK_y y} \tag{100}$$

Considering the first term as incident field E_i and the second term as reflected field E_r, we see that the reflection coefficient at the boundary $(y = -\frac{c}{2})$ is:

$$R_I = \frac{E_r}{E_i} \Big|_{y=-\frac{c}{2}} = \pm \frac{e^{iK_y(b+c)} e^{-iK_y \frac{c}{2}}}{e^{iK_y \frac{c}{2}}} = \pm e^{iK_y b} \quad , \tag{101}$$

where the upper sign corresponds to even modes and the lower sign to odd modes.

In guide II, the incident and reflected electric fields at the boundary $(y = \frac{c}{2})$ are $B'' e^{iK_y y}$ and $e^{-iK_y y}$, respectively.

The reflection coefficient at this boundary is given by:

$$R_{II} = \frac{E_r}{E_i} \Big|_{y=\frac{c}{2}} = \frac{e^{iK_y y}}{\pm e^{-iK_y(b+c)} e^{-iK_y y}} \Big|_{y=\frac{c}{2}} = \pm e^{iK_y b} \tag{102}$$

where the upper isgn corresponds to even modes and the lower sign to odd modes. We see that $R_I = R_{II} = R$; this result is due to the fact that guides I and II are symmetric. Also, we observe that:

$$|R|^2 = |\pm e^{iK_y b}|^2 = 1 \quad , \tag{103}$$

which indicates a total internal reflection.

In the region $(0 \leq z \leq L)$ the situation is totally different. Here both waveguides are present and there is power coupling between them; the reflection coefficients at the boundaries $(y = -\frac{c}{2})$ and $(y = \frac{c}{2})$ are smaller than R in absolute value. To improve the boundary conditions, we have to consider

the system of two waveguides. The eigenvalue equations for modes and the propagation constants are different from those pertaining to regions $(z \leq 0)$ and $(z \leq L)$.

CHAPTER III

TWO PARALLEL RECTANGULAR WAVEGUIDES

In this chapter we will derive the eigenvalue equation for modes in the region ($0 \leq z \leq L$). We will then present the method of Miller and Marcatili for the solution of coupling between the two waveguides in this region, and discuss the validity of their approximations.

The reflection and transmission coefficients at the boundaries ($y = -\frac{c}{2}$) and ($y = \frac{c}{2}$) are important parameters to find the eigenvalue equations for modes and to find the coupling between the two guides.

In the case of the single rectangular waveguide studied in Chapter II, let us take a close look at the eigenvalue equation (55). We see that the condition for guided modes can be stated as follows:

$$2 i K_y b = 2 i \pi n \, , + \text{ the phase of the reflection coefficient at}$$
$$\text{the boundary } (y = -\frac{c}{2})$$
$$+ \text{ the phase of the reflection coefficient at}$$
$$\text{the boundary } (y = \frac{c}{2}).$$

To find the eigenvalue equation for modes for guides I and II in the region ($0 \leq z \leq L$), we will use the same method already used in Chapter II. For guide I the reflection coefficient at the boundary ($y = -\frac{c}{2} - b$) is the same as in region ($z \leq 0$), but because of the presence of guide II the reflection coefficient at the boundary ($y = -\frac{c}{2}$) is different from its value in the region ($z \leq 0$).

3.1 <u>Reflection and Transmission Coefficients in the Region ($0 \leq z \leq L$) at the Inner Boundaries ($y = -\frac{c}{2}$) and ($y = \frac{c}{2}$).</u>

The two waveguides are identical, so the reflection coefficient in guide I at the boundary ($y = -\frac{c}{2}$) is the same as the reflection coefficient in guide II at the boundary ($y = \frac{c}{2}$).

In Fig. 3 we assume that E_i and E_r be the incident and reflected electric fields at the boundary $(y = -\frac{c}{2})$, and that E_t by the transmitted electric field. The arrows indicate the direction of propagation (normal to the wavefront).

Inside guide I, the electric and magnetic fields can be written as:

$$E = E_i + E_r = E_i e^{-iK_y(y+\frac{c}{2})} + E_r e^{iK_y(y+\frac{c}{2})} \quad , \tag{104}$$

$$H = H_i + H_r = -\frac{\omega\epsilon_0 K_y}{k^2}\left[E_i e^{-iK_y(y+\frac{c}{2})} - E_r e^{iK_y(y+\frac{c}{2})}\right] \; ; \tag{105}$$

in the region $(-\frac{c}{2} \leq y \leq +\frac{c}{2})$ we have:

$$E = E_1 e^{-\gamma_s(y+\frac{c}{2})} + E_2 e^{\gamma_s(y+\frac{c}{2})} \quad , \tag{106}$$

$$H = \frac{i\omega\epsilon_0\gamma_s}{k^2}\left[E_1 e^{-\gamma_s(y+\frac{c}{2})} - E_2 e^{\gamma_s(y+\frac{c}{2})}\right] \; ; \tag{107}$$

in the region $(y \geq \frac{c}{2})$ inside guide II we have:

$$E = E_t e^{-iK_y(y-\frac{c}{2})} \quad , \tag{108}$$

$$H = -\frac{\omega\epsilon_0 K_y}{k^2} E_t e^{-iK_y(y-\frac{c}{2})} \quad . \tag{109}$$

The boundary conditions are:

at $(y = -\frac{c}{2})$:
$$E_i + E_r = E_1 + E_2 \tag{110}$$

$$E_i - E_r = -i\frac{\gamma_s}{K_y}(E_1 - E_2) \tag{111}$$

at $y = \frac{c}{2}$:

$$E_1 e^{-\gamma_5 c} + E_2 e^{\gamma_5 c} = E_t \tag{112}$$

$$E_1 e^{-\gamma_5 c} - E_2 e^{\gamma_5 c} = i \frac{K_y}{\gamma_5} E_t \tag{113}$$

Therefore, by dividing eq. (113) by (112):

$$i \frac{K_y}{\gamma_5} = \frac{E_1 e^{-\gamma_5 c} - E_2 e^{\gamma_5 c}}{E_1 e^{-\gamma_5 c} + E_2 e^{\gamma_5 c}} \tag{114}$$

From (114) we find that:

$$\Gamma = \frac{E_2}{E_1} = e^{-2\gamma_5 c} \times \frac{1 - i \frac{K_y}{\gamma_5}}{1 + i \frac{K_y}{\gamma_5}} = -e^{-2\gamma_5 c + 2i \arctan \frac{\gamma_5}{K_y}} \tag{115}$$

$$= e^{-2\gamma_5 c} \times \frac{\gamma_5 - i K_y}{\gamma_5 + i K_y}$$

and, therefore, from (110) and (111):

$$\frac{E_i - E_r}{E_i + E_r} = \frac{i \gamma_5}{K_y} \times \frac{E_1 - E_2}{E_1 + E_2} = -i \frac{\gamma_5}{K_y} \cdot \frac{1 - \Gamma}{1 + \Gamma} \tag{116}$$

The quantity $R' = \frac{E_r}{E_i}$ is the reflection coefficient; from (116) we find:

$$\frac{1 - R'}{1 + R'} = -i \frac{\gamma_5}{K_y} \left[1 - \frac{\gamma_5 - i K_y}{\gamma_5 + i K_y} e^{-2\gamma_5 c} \right] \Big/ \left[1 + \frac{\gamma_5 - i K_y}{\gamma_5 + i K_y} e^{-2\gamma_5 c} \right] \tag{117}$$

$$= -i \frac{\gamma_5}{K_y} \cdot \frac{\gamma_5 + i K_y - (\gamma_5 - i K_y) e^{-2\gamma_5 c}}{\gamma_5 + i K_y + (\gamma_5 - i K_y) e^{-2\gamma_5 c}} = -i \frac{\gamma_5}{K_y} \cdot \frac{\gamma_5 \, Sh \gamma_5 c + i K_y \, Ch \gamma_5 c}{\gamma_5 \, Ch \gamma_5 c + i K_y \, Sh \gamma_5 c} \tag{118}$$

$$= \frac{\gamma_5 K_y \, Ch \gamma_5 c - i \gamma_5^2 \, Sh \gamma_5 c}{\gamma_5 K_y \, Ch \gamma_5 c + i K_y^2 \, Sh \gamma_5 c} \tag{119}$$

or:
$$R' = \frac{i(\gamma_5^2 + K_y^2) \, Sh \, \gamma_5 c}{2\gamma_5 K_y Ch \, \gamma_5 c + i(K_y^2 - \gamma_5^2) \, Sh \, \gamma_5 c} \tag{120}$$

$$R' = \frac{1}{\frac{K_y^2 - \gamma_5^2}{K_y^2 + \gamma_5^2} - i \, \frac{2\gamma_5 K_y}{K_y^2 + \gamma_5^2} \, coth \, \gamma_5 c} \tag{121}$$

$$= (K_y^2 + \gamma_5^2) \, e^{i \, arctan\left(\frac{2K_y \gamma_5}{K_y^2 - \gamma_5^2} coth \, \gamma_5 c\right)} \Bigg/ \sqrt{(K_y^2 - \gamma_5^2)^2 + 4K_y^2 \gamma_5^2 \, coth^2 \gamma_5 c}$$

$$R' = \left[1 + \frac{4 K_y^2 \gamma_5^2}{(K_y^2 + \gamma_5^2) \, Sh^2 \gamma_5 c}\right]^{-\frac{1}{2}} e^{i \, arctan\left(\frac{2K_y \gamma_5}{K_y^2 - \gamma_5^2} coth \, \gamma_5 c\right)} . \tag{122}$$

We put:

$$u = \frac{Sh \, \gamma_5 c}{2 K_y \gamma_5 / (K_y^2 + \gamma_5^2)} \tag{123}$$

and finally we find:

$$R' = \frac{u}{\sqrt{1 + u^2}} \, e^{i \, arctan\left(\frac{2K_y \gamma_5}{K_y^2 - \gamma_5^2} coth \, \gamma_5 c\right)} \tag{124}$$

The transmission coefficient is: $t = \dfrac{E_t}{E_i}$.

From (112) and (113) we find:

$$E_t = \frac{2E_i \, e^{-\gamma_5 c}}{1 + i \, \frac{K_y}{\gamma_5}} \quad ; \tag{125}$$

from (110) and (111) we have:

$$2E_1 = E_i\left(1 + i\frac{\gamma_5}{K_y}\right) + E_r\left(1 - i\frac{\gamma_5}{K_y}\right) .$$

(126)

From (125) and (126) and using $\dfrac{E_r}{E_i} = R'$ we find:

$$t = e^{-\gamma_5 c}\left(1 + R' \frac{1 - i\dfrac{K_y}{\gamma_5}}{1 + i\dfrac{K_y}{\gamma_5}}\right) .$$

(127)

Substituting R' from (120) in (127) we find:

$$t = e^{-\gamma_5 c}\left\{1 + \frac{i(\gamma_5^2 + K_y^2)\,Sh\,\gamma_5 c\,(\gamma_5 - iK_y)}{\left[2\gamma_5 K_y Ch\,\gamma_5 c + i(K_y^2 - \gamma_5^2)Sh\,\gamma_5 c\right](\gamma_5 + iK_y)}\right\}$$

(128)

$$t = e^{-\gamma_5 c}\left\{\left[2\gamma_5^2 K_y Ch\,\gamma_5 c - K_y(K_y^2 - \gamma_5^2)Sh\,\gamma_5 c + K_y(K_y^2 + \gamma_5^2)Sh\,\gamma_5 c\right] + \right.$$

$$\left. i\left[\gamma_5(K_y^2 - \gamma_5^2)Sh\,\gamma_5 c + 2\gamma_5 K_y^2 Ch\,\gamma_5 c + \gamma_5(K_y^2 + \gamma_5^2)Sh\,\gamma_5 c\right]\right\} \times$$

$$\left\{\left[2\gamma_5 K_y Ch\,\gamma_5 c + i(K_y^2 - \gamma_5^2)Sh\,\gamma_5 c\right](\gamma_5 + iK_y)\right\}^{-1}$$

(129)

$$t = e^{-\gamma_5 c}\left[2K_y\gamma_5^2(Ch\,\gamma_5 c + Sh\,\gamma_5 c) + 2i\gamma_5 K_y^2(Ch\,\gamma_5 c + Sh\,\gamma_5 c)\right] \times$$

$$\left\{(\gamma_5 + iK_y)\left[2\gamma_5 K_y Ch\,\gamma_5 c + i(K_y^2 - \gamma_5^2)Sh\,\gamma_5 c\right]\right\}^{-1}$$

(130)

Now substitute $ch\gamma_5 c + sh\gamma_5 c = e^{\gamma_5 c}$ in (130); we find:

$$t = \frac{2K_y \gamma_5}{2\gamma_5 K_y Ch\gamma_5 C + i(K_y^2 - \gamma_5^2)Sh\gamma_5 c} \quad .$$

(131)

Comparing (131) with (120), we find:

$$t = \frac{2K_y \gamma_5}{i(K_y^2 + \gamma_5^2)Sh\gamma_5 c} R' = -\frac{i}{u}R'$$

(132)

$$= -\frac{i}{\sqrt{1+u^2}} e^{i \arctan\left(\frac{2K_y\gamma_5}{K_y^2-\gamma_5^2}Coth\gamma_5 c\right)}$$

We see that:

$$|R'|^2 + |t|^2 = \frac{u^2}{1+u^2} + \frac{1}{1+u^2} = 1$$

so that energy is conserved.

We now consider two limiting cases.

a) For infinitely large separation distance we have:

$$u = \infty \quad \text{and} \quad \cotanh \gamma_5 c = 1,$$

(133)

and from (124) and (132), we have:

$$R' = e^{i \arctan\left(\frac{2K_y\gamma_5}{K_y^2-\gamma_5^2}\right)} = e^{iK_y b} = R$$

(134)

$$t = 0.$$

b) For zero separation distance (c = 0), the two guides touch each
other and form one guide with a width equal to twice the width of
the original guide. In this case we have:

$$u = 0 \quad \text{and} \quad \coth \gamma_5 c = \infty ;$$

(135)

substituting in (124) and (125), we find:

$$R' = 0$$

and:

$$t = -\frac{i}{\sqrt{1+0}} \, e^{i \arctan \infty} = -i e^{i \cdot \frac{\pi}{2}} = 1 \tag{136}$$

An important remark: We see that at any given distance c we have a phase difference of $\pi/2$ between reflection and transmission coefficients.

3.2 Guided Modes of the System of Two Guides in Region ($0 \leq z \leq L$).

The condition for guided modes in guide I is the following:

$2 i K_y b$ = the phase of the reflection coefficient at the boundary
$(y = -b - \frac{c}{2})$ + the phase of the reflection coefficient
at the boundary $(y = -\frac{c}{2})$ + $2in\pi$

= phase of R + phase of R' + $2in\pi$

$$= i \arctan \frac{2\gamma_5 K_y}{K_y^2 - \gamma_5^2} + i \arctan \left(\frac{2\gamma_5 K_y}{K_y^2 - \gamma_5^2} \coth \gamma_5 c \right) + 2in\pi, \tag{137}$$

or:

$$\tan 2K_y b = \left[\frac{2K_y \gamma_5}{K_y^2 - \gamma_5^2} + \frac{2K_y \gamma_5}{K_y^2 - \gamma_5^2} \coth \gamma_5 c \right] \times \left[1 - \left(\frac{2K_y \gamma_5}{K_y^2 - \gamma_5^2} \right)^2 \coth \gamma_5 c \right]^{-1} \tag{138}$$

$$= F(K_y b)$$

The solution of this eigenvalue equation determines the propagation constants of the propagating modes in the region ($0 \leq z \leq L$). In Fig. 5, we plotted $\tan 2K_y b$ and $F(K_y b)$ in function of the variable $x = K_y b$ and for
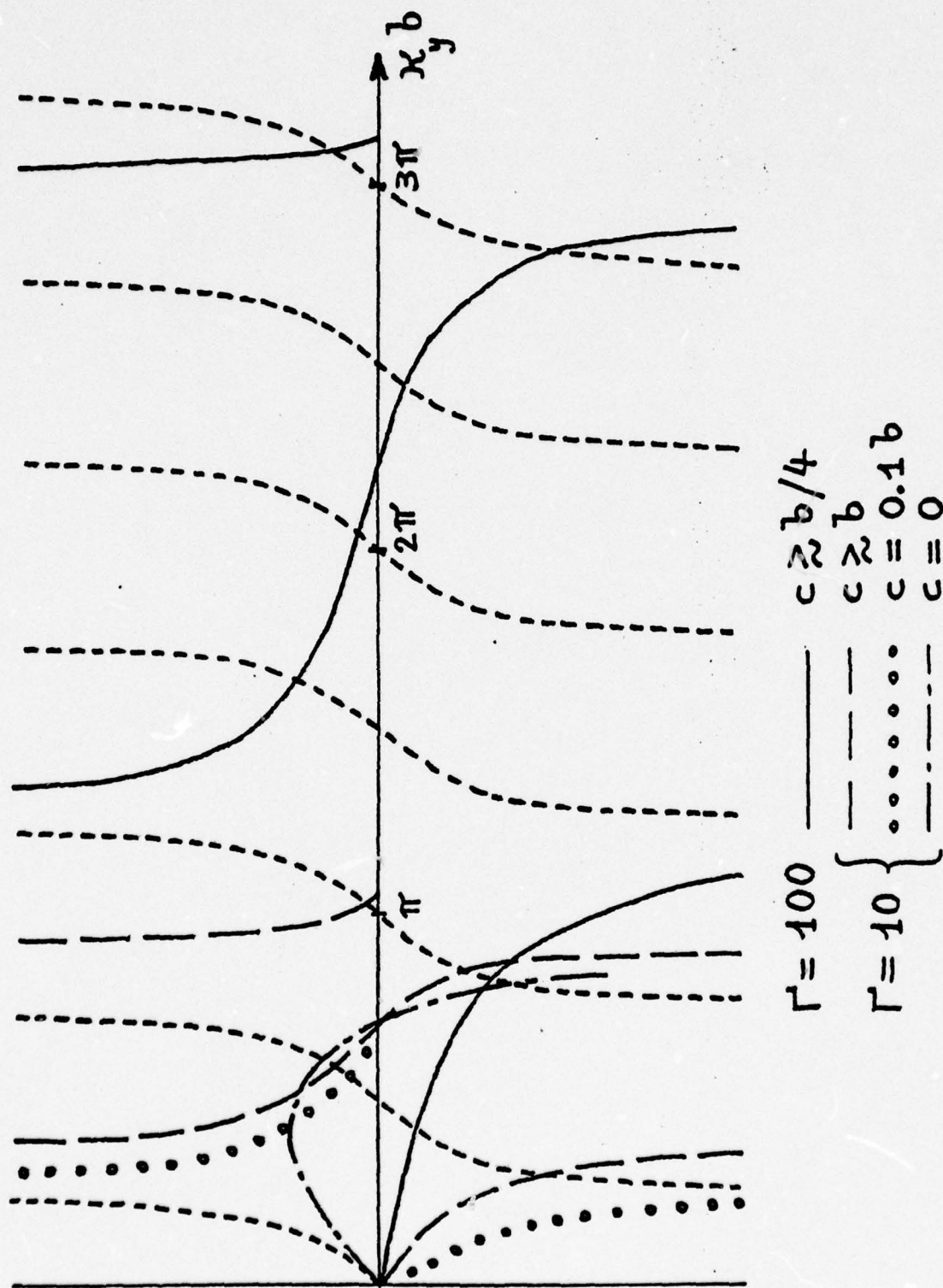
Figure 5. Eigenvalues for system of two coupled waveguides, for different values of $\Gamma = (n_1^2 - n_2^2)k^2b^2$ and for different separation distance c.

different values of the distance c. Each intersection of the two above functions determines a mode and a propagation constant K_y.

We see that for $(n_1^2 - n_2^2)k^2b^2 = 100$ we have eight modes, and for $(n_1^2 - n_2^2)k^2b^2 = 10$ we have four modes. The modes are cutoff when $K_y \geq k\sqrt{n_1^2 - n_2^2}$. By comparing Fig. 4 with Fig 5, we see that the propagation constant K_y varies very slowly with the separation distance c. This means that, except when c is quite small, an incident mode propagates approximately unchanged into the coupling region. In other words, the coupling coefficients of the incident mode to the modes that can exist in the coupling region are extremely small except for the mode whose K_y is almost identical to the K_y of the incident mode (in this case, the coupling coefficient is very close to unity). For example, if $(n_1^2 - n_2^2)k^2b^2 = 100$ or 10, then the above statements are true for all $c \geq b/4$ or $c \geq b$, respectively.

In the region $(0 \leq z \leq L)$, we have exchange of power between the two guides, which means that the power in each guide is a function of z. With the assumption that the x- and y-dependence along the z-axis does not change, the power in each guide is proportional to the square of the absolute value of the field amplitude. The power P_I will be proportional to $|A|^2$ and P_{II} will be proportional to $|B|^2$. Our problem is to find the z-dependence of the amplitudes A and B.

Dropping the x- and y-dependence, the electric field can be written as:

$$E_1 = A(z)\, e^{-i\beta z}, \text{ in guide I,} \tag{139}$$

$$E_2 = B(z)\, e^{-i\beta z}, \text{ in guide II,} \tag{140}$$

where β is the propagation constant in the z-direction in region $(0 \leq z \leq L)$.

3.3 Miller-Marcatili Solution.

S. E. Miller [2] studied the coupling between transmission lines. His solution was used by Marcatili [3] and applied to the coupling between rectangular waveguides.

Miller set up two differential equations for the field amplitudes in guide I and guide II:

$$\frac{\delta E_1}{\delta z} = -(\Gamma_1 + k_{11}) E_1 + k_{21} E_2 \quad , \tag{1'}$$

$$\frac{\delta E_2}{\delta z} = k_{21} E_1 - (\Gamma_2 + k_{22}) E_2 \quad , \tag{2'}$$

where $\frac{\delta E_1}{\delta z}$ and $\frac{\delta E_2}{\delta z}$ are the rates of change in the electric field in guide I and guide II.

If the two waveguides are identical we have that $\Gamma_1 = \Gamma_2 = \Gamma = i\beta$, the propagation constant in the z-direction in the coupling region; $k_{11} = k_{22}$ represents the reaction of the coupling mechanism on the guide itself, and $k_{21} = k_{12} = k$ represents the transfer effect of the coupling mechanism.

Miller assumes a very large separation distance between the lines; this means, in our case, that the change in the propagatio- constants between the regions $(0 \leq z)$ and $(0 \leq z \leq L)$ is negligible and the coupling between the two guides is very weak so that no reflected wave is excited. Also, Miller assumes that $k = k_{12} = k_{21}$ is a purely imaginary quantity, which is a necessary condition to have a solution to equations (1') and (2') in the form:

$$E_1 = \cos Cz \ e^{-(iC+\gamma)z}, \tag{3'}$$

$$E_2 = i \sin Cz \ e^{-(iC+\gamma)z}. \tag{4'}$$

Here E_1 and E_2 are normalized. $C = -ik$ is the coupling coefficient, and $\beta_0 = -i\gamma$ is the propagation constant in the z-direction in the region $(0 \geq z)$.

Marcatili [3] considers C as the change in the propagation constant in the z-direction between the two regions $(0 \leq z)$ and $(0 \leq z \leq L)$, and he calculates C by solving Maxwell's equations in the region $(0 \leq z \leq L)$ for

the system of two parallel waveguides. By imposing the boundary conditions, he finds the propagation constant β, then:

$$C = \beta - \beta_0 = -ik = \frac{2K_y^2 \gamma_5}{\beta_0 b} \cdot \frac{e^{-\gamma_5 c}}{K_y^2 + \gamma_5^2} \quad . \tag{5$'$}$$

The value of the distance L necessary for a complete transfer of power from guide I to guide II corresponds to:

$$|E_2| = 1 = \sin\frac{\pi}{2} \tag{6$'$}$$

or

$$\frac{\pi}{2} = CL \quad \text{and} \quad L = \frac{\pi}{2C} = \frac{\pi \beta b \ (K_y^2 + \gamma_5^2) \ e^{\gamma_5 c}}{4K_y^2 \gamma_5} . \tag{7$'$}$$

Marcatili points out that his solution is valid only for a very large separation distance c, for which the distance L given in (7$'$) equals several hundred wavelengths.

We want to investigate the validity of Miller's solution. He implies that k is a purely imaginary quantity; if we investigate this quantity closely, we see that it is given by the relation:

$$k = \frac{K_y}{2\beta b} t \ e^{i\phi}, \tag{8$'$}$$

where t is the transmission coefficient and ϕ is the phase change when the optical ray travels the path between two successive incidences. ϕ cannot be known unless we know the z-dependence of the field components.

In our method, we use real quantities in setting up differential equations, and these are known without imposing a type of solution in advance. We show that Miller's assumption that k is purely imaginary is not valid. Our solution will remain valid for a small separation distance, and therefore strong coupling, between the guides.

CHAPTER IV

COUPLING BETWEEN RECTANGULAR WAVEGUIDES

4.1 <u>Coupling Mechanism</u>.

In Fig. 3 we represent the system of two parallel identical rectangular waveguides in the y-z plane. We assume the two guides to be extended from $(z = -\infty)$ to $(z = +\infty)$, with only one excited mode traveling in the positive z-direction in both guides.

In guide I we will follow the path of an optical ray with associated electric field E_1. This ray is partially reflected at the interface $(y = -\frac{c}{2})$ at the point $z = z_1$, and partially transmitted to guide II. Also at the same point z, in guide II we have an incident electric field $\vec{E}_2$ at the interface $(y = +\frac{c}{2})$. We can write the two fields $\vec{E}_1{'}$ and $\vec{E}_2{'}$ at the point z_1, in terms of $\vec{E}_1$ and $\vec{E}_2$ at the same point as follows:

$$\vec{E}_1{'} = R{'}\vec{E}_1 + t\vec{E}_2 \tag{141}$$

$$\vec{E}_2{'} = R{'}\vec{E}_2 + t\vec{E}_1 \tag{142}$$

$R{'}$ and t are the reflection and transmission coefficients at the boundaries $(y = -\frac{c}{2})$ and $(y = \frac{c}{2})$; they are the same for both guides.

The fields $\vec{E}_1{'}$ and $\vec{E}_2{'}$ are totally reflected at the boundaries $(y = -(b + \frac{c}{2}))$ and $(y = b + \frac{c}{2})$, then they meet again at the coordinate z_2; at this point the two fields can be written as:

$$\vec{E}_1{'}\Big|_{z=z_2} = \vec{E}_1{'}\Big|_{z=z_1} e^{i\phi} = (R{'}\vec{E}_1 + t\vec{E}_2) e^{i\phi} \tag{143}$$

$$\vec{E}_2{'}\Big|_{z=z_2} = \vec{E}_2{'}\Big|_{z=z_1} e^{i\phi} = (R{'}\vec{E}_2 + t\vec{E}_1) e^{i\phi} \tag{144}$$

where ϕ is a constant phase. $\vec{E}_1$ and $\vec{E}_2$ are the incident electric fields in guide I and guide II respectively at the point z_1, while $\vec{E}_1{}'|_{z=z_2}$ and $\vec{E}_2{}'|_{z=z_2}$ are the fields obtained after traveling one bounce along the z-axis. We have that

$$\Delta z = z_2 - z_1 = \frac{2\beta b}{K_y}. \tag{145}$$

Firstly, we impose energy conservation. We know that the power in guide I is proportional to $|E_1|^2$ and the power in guide II is proportional to $|E_2|^2$. With E_1 and E_2 normalized, the total power will be:

$$P_0 = P_I + P_{II} = |E_1|^2 + |E_2|^2. \tag{146}$$

We want to verify that $|E_1(z_2)|^2 + |E_2(z_2)|^2 = |E_1(z_1)|^2 + |E_2(z_1)|^2$. We can always write:

$$E_2 = a\,e^{i\theta}E \tag{147}$$

with $a(z)$ and $\theta(z)$ functions of z.

Substituting (147) in (141) and (142), we find:

$$E_1(z_2) = (R'E_1 + t\,a\,e^{i\theta}E_1)\,e^{i\theta} = E_1\,e^{i\theta}(R' + t\,a\,e^{i\theta}). \tag{148}$$

$$E_2(z_2) = (R'a\,e^{i\theta}E_1 + t\,E_1)\,e^{i\theta} = E_1\,e^{i\theta}(R'a\,e^{i\theta} + t). \tag{149}$$

and

$$|E_1(z_2)|^2 + |E_2(z_2)|^2 = |E_1|^2\,(|R' + t\,a\,e^{i\theta}|^2 + |R'a\,e^{i\theta} + t|^2) \tag{150}$$

Substituting R' and t from (124) and (132), eq. (150) becomes:

$$|E_1(z_2)|^2 + |E_2(z)|^2 = \frac{|E_1(z_1)|^2}{1+u^2}\left[\,|u + i\,a\,e^{i\theta}|^2 + |a\,u\,e^{i\theta} - i|^2\,\right]$$

$$= \frac{|E_1|^2}{1+u^2}\left[(u + a\sin\theta)^2 + a^2\cos^2\theta + a^2 u^2\cos^2\theta + (a\,u\sin\theta - 1)^2\right]$$

$$= \frac{|E_1|^2}{1+u^2}\left(u^2 + 2a\,u\sin\theta + a^2\sin^2\theta + a^2\cos^2\theta + a^2 u^2\cos^2\theta + a^2 u^2\sin^2\theta - 2a\,u\sin\theta + 1\right)$$

Finally,

$$P_0(z_2) = |E_1(z_2)|^2 + |E_2(z_2)|^2 = \frac{|E_1|^2}{1+u^2}(1+u^2)(1+a^2) = |E_1(z_1)|^2 + |E_2(z_1)|^2$$

$$P_0(z_2) = P_0(z_1)$$

Thus, we see that the total power is constant along z.

4.2 Differential Equations for Power Transfer.

In calculating the power in each guide as a function of z, we deal with the absolute values of the field components and can thus eliminate some of the phases involved, particularly the phase ϕ which is as yet unknown.

We assume that the bounce length $\Delta z = z_2 - z_1$ is very small compared to the distance L, and that the separation distance c is large enough so that we can consider the power exchange per bounce as a very small quantity. Also, we assume that the propagation constants in the three regions $(0 \geq z)$, $(0 \leq z \leq L)$ and $(z \geq L)$ are the same, so that there is no reflection at the boundary $(z = 0)$ in guide II, and we neglect the reflected wave in guide I at the boundary $(z = L)$. This last assumption means that either the power P_I is negligible at $(z = L)$, or guide I is tapered about $(z = L)$, or both.

The field amplitudes in guide I and guide II are called A and B respectively, as in Chapter II. After dropping the x- and y-dependences, which are the same as in region $(z \leq 0)$ for guide I and as in region $(z \geq L)$ for guide II, the fields in region $(0 \leq z \leq L)$ are given by:

$$E_1 = A(z)\, e^{i\beta z}; \tag{151}$$

$$E_2 = B(z)\, e^{i\beta z}; \tag{152}$$

A and B are functions of z because of the coupling between the two guides. Here we consider the normalized value of A and B and we have the following relations:

$$P_I(z) = |A(z)|^2, \tag{153}$$

$$P_{II}(z) = |B(z)|^2. \tag{154}$$

The change of P_I per unit length along z is given by:

$$\frac{dP_I}{dz} = \frac{P_I(z_2) - P_I(z_1)}{z_2 - z_1} = \frac{K_y}{2\beta b}\left(|A(z_2)|^2 - |A(z_1)|^2\right) \tag{155}$$

$$= \frac{K_y}{2\beta b}\left(|R'A(z_1) + tB(z_1)|^2 |e^{i\phi}|^2 - |A(z_1)|^2\right) \tag{156}$$

We can write:

$$B(z) = a(z)\, e^{i\theta(z)}\, A(z) \tag{157}$$

substitute (155), (124) and (132) in (154), and find:

$$\frac{dP_I}{dz} = \frac{K_y}{2\beta b}|A(z_1)|^2\left(\frac{1}{1+u^2}|u - i\,\alpha e^{i\theta}|^2 - 1\right) \tag{158}$$

$$= \frac{K_y}{2\beta b}P_I\left(\frac{u^2}{1+u^2} + \frac{2u a \sin\theta}{1+u^2} + \frac{a^2\sin^2\theta}{1+u^2} + \frac{a^2\cos^2\theta}{1+u^2} - 1\right) \tag{159}$$

$$= \frac{K_y}{2\beta b(1+u^2)}P_I\left(-1 + a^2 + 2u a \sin\theta\right) = D\left[-P_I + a^2 P_I + 2u a \sin\theta\, P_I\right] \tag{160}$$

$$\frac{dP_I}{dz} = D\left[-P_I + P_{II} + 2u\sqrt{P_I P_{II}}\,\sin\theta\right]$$

where the constant $D = \dfrac{K_y}{2\beta b(1+u^2)}$. For P_{II} we have:

$$\frac{dP_{II}}{dz} = \frac{K_y}{2\beta b}\left(|B(z_2)|^2 - |B(z_1)|^2\right) = \frac{K_y}{2\beta b}\left[|BR' + tA|^2 - |B|^2\right]$$

$$= D|A|^2\left(|uae^{i\phi} - i|^2 - a^2\right)$$

$$= D|A|^2\left[u^2a^2\cos^2\theta + u^2a^2\sin^2\theta + 2ua\sin\theta + 1 - a^2(1+u^2)\right]$$

(161)

$$\frac{dP_{II}}{dz} = D|A|^2\left(-a^2 + 1 - 2au\sin\theta\right) = D\left(-P_{II} + P_I - 2u\sqrt{P_I P_{II}}\sin\theta\right)$$

(162)

We see that $\dfrac{dP_{II}}{dz} + \dfrac{dP_I}{dz} = 0$, so that energy is conserved. To solve the two differential equations (160) and (162), we must find the function $\sin\theta(z)$, where $\theta(z)$ is the phase difference between B and A. We write:

$$\frac{d\theta(z)}{dz} = \frac{K_y}{2\beta b}\left[\theta(z_2) - \theta(z_1)\right] ;$$

(163)

$\theta(z_1)$ is the phase difference between $B(z_1)$ and $A(z_1)$, and $\theta(z_2)$ is the phase difference between $B(z_2)$ and $A(z_2)$.

$$\theta(z_2) = \text{phase of } B(z_2) - \text{phase of } A(z_2)$$

$$= \text{phase}\left[R'B(z_1) + tA(z_1)\right]e^{i\phi} - \text{phase}\left[R'A(z_1) + tB(z_1)\right]e^{i\phi}$$

$$= \text{phase}\left(ae^{i\theta} - \frac{i}{u}\right)A(z_1)e^{i\arctan\left(\frac{2K_y\gamma_s}{K_y^2 - \gamma_s^2}\coth\gamma_s c\right)}$$

$$\quad - \text{phase}\left(1 - \frac{ia}{u}e^{i\theta}\right)A(z_1)e^{i\arctan\left(\frac{2K_y\gamma_s}{K_y^2 - \gamma_s^2}\coth\gamma_s c\right)}$$

$$= \text{phase}\left[a\cos\theta + i(a\sin\theta - \frac{1}{u})\right] - \text{phase}\left[1 + \frac{a\sin\theta}{u} - i\frac{a\cos\theta}{u}\right]$$

$$\theta(z_2) = \arctan\frac{a\sin\theta - \frac{1}{u}}{a\cos\theta} - \arctan\frac{-a\cos\theta}{u(1 + \frac{a\sin\theta}{u})}$$

(164)

therefore:

$$\tan \theta(z_2) = \left[\frac{a \sin\theta - \frac{1}{u}}{a\cos\theta} + \frac{\frac{a}{u}\cos\theta}{1+\frac{a}{u}\sin\theta} \right] \Big/ \left[1 - \frac{a\sin\theta - \frac{1}{u}}{a\cos\theta} \cdot \frac{\frac{a}{u}\cos\theta}{1+\frac{a}{u}\sin\theta} \right]$$

$$= \frac{u^2-1}{u^2+1} \tan\theta - \frac{u(1-a^2)}{a(u^2+1)} \cdot \frac{1}{\cos\theta} \tag{165}$$

and

$$\frac{d\tan\theta}{dz} = \frac{\Delta\tan\theta}{\Delta z} = \frac{\tan\theta(z_2) - \tan\theta(z_1)}{z_2 - z_1} = \frac{K_y}{2\beta b(1+u^2)} \left(-2\tan\theta - u\frac{1-a^2}{a\cos\theta} \right). \tag{166}$$

Finally, substituting $D = \dfrac{K_y}{2\beta b(1+u^2)}$ and $\dfrac{P_{II}}{P_I} = a^2$, we find:

$$\frac{d\sin\theta}{dz} = -D\cos^2\theta \left(2\sin\theta + u\frac{P_I - P_{II}}{\sqrt{P_I P_{II}}} \right)$$

$$= -D(1-\sin^2\theta)\left(2\sin\theta + u\frac{P_I P_{II}}{\sqrt{P_I P_{II}}} \right) \tag{167}$$

Eqs. (160), (162) and (167) are three differential equations, which are to be solved simultaneously. These equations represent our general formulation of the coupling between two waveguides.

4.3 Particular Cases.

We discuss two particular cases.

a) Assume that at an arbitrary point z it happens that the fields in the two waveguides are the same in phase and magnitude; then A ≡ B. At the next incidence, i.e. after one bounce, we have that

$$A(z + \Delta z) = [A(z)R^{\prime} + A(z)t]\ e^{i\phi}, \tag{168}$$

$$B(z + \Delta z) = [A(z)R^{\prime} + A(z)t]\ e^{i\phi}. \tag{169}$$

The two fields have the same phase and same magnitude, and the waves propagate in the two waveguides as if there were no coupling between them.

b) This case applies to the geometry of Fig. 3. We assume that at a given point z in the field amplitude B is leading A by $\frac{3\pi}{2}$: then at the point z + Δz we have:

$$A(z + \Delta z) = [R^{\prime}A(z) + tB(z)]\ e^{i\phi}, \tag{170}$$

$$B(z + \Delta z) = [R^{\prime}B(z) + tA(z)]\ e^{i\phi}. \tag{171}$$

or:

$$A(z+\Delta z) = \left[R'A(z) - \frac{iR'a}{u}e^{i\frac{3\pi}{2}}A(z)\right]e^{i\phi} = R'A(z)e^{i\phi}\left(1-\frac{a}{u}\right) \tag{172}$$

$$B(z+\Delta z) = \left[R'a\,A(z)e^{i\frac{3\pi}{2}} - i\frac{R'}{u}A(z)\right]e^{i\phi} = -iR'A(z)e^{i\phi}\left(a+\frac{1}{u}\right)$$

$$= R'A(z)e^{i\phi}\left(a+\frac{1}{u}\right)e^{i\frac{3\pi}{2}} \tag{173}$$

From (172) and (173) we see that the magnitude of A decreases, and the magnitude of B increases but the phase difference remains the same, i.e. $\frac{3\pi}{2}$, which means that the phase $\theta(z)$ remains constant along z, and

$$\sin\theta = -1. \tag{174}$$

An important observation is that when the transmitted amplitude from guide I to guide II and the reflected amplitude in guide II are in phase, the total amplitude is the sum of their absolute values. But if the transmitted amplitude from guide II to guide I and the reflected amplitude in guide I have a phase difference of π, then

the resulting total amplitude is the difference between their
absolute values.

Substituting (174) in (160) and (162), we find the differential
equations for the coupling between the two rectangular waveguides
in the form:

$$\frac{dP_I}{dz} = -D\left[P_I - P_{II} + 2u\sqrt{P_I P_{II}} \right] \tag{175}$$

$$\frac{dP_{II}}{dz} = D\left[P_I - P_{II} + 2u\sqrt{P_I P_{II}} \right] \tag{176}$$

These two differential equations are valid as long as the phase of
the field in guide II is leading the phase of the field in guide I
by $\frac{3\pi}{2}$. However, when the magnitude of A is so small that the trans-
mitted field from guide II to guide I is larger in magnitude than
the reflected field in guide I, then the phase of the resultant
field in guide I will lead the phase of the field in guide II. In
this later situation, the power begins to flow back from guide II
to guide I, and eq. (174) becomes:

$$\sin \theta = +1 \tag{177}$$

Substituting (177) in (160) and (162), we get the differential
equations

$$\frac{dP_I}{dz} = D\left[P_{II} - P_I + 2u\sqrt{P_I P_{II}} \right] \;, \tag{178}$$

$$\frac{dP_{II}}{dz} = -D\left[P_{II} - P_I + 2u\sqrt{P_I P_{II}} \right] \;, \tag{179}$$

which differ from (175) and (176).

The change in the phase θ from $\frac{3\pi}{2}$ to $-\frac{3\pi}{2}$ occurs when:

$$|R'A| \leq |tB| \quad \text{or} \quad |R'A|^2 \leq |tB|^2 \quad \text{or} \quad |R'|^2 P_I \leq |t|^2 P_{II}. \tag{180}$$

Substituting (146) in (180) we get:

$$\frac{u^2}{1+u^2} P_I \leq \frac{1}{1+u^2}(P_0 - P_I) \tag{181}$$

or:

$$P_{II} \geqslant P_0 - \frac{P_0}{1+u^2} = \frac{u^2}{1+u^2} P_0 \,, \quad \text{and} \quad P_I \leqslant \frac{P_0}{1+u^2} \tag{182}$$

4.4 Solution for Power Transfer.

4.4.1 Approximate Solution.

In these two differential equations, let us compare $P_I - P_{II}$ to $2u\sqrt{P_I P_{II}}$, with the first bounce after $z = 0$. We have:

$$P_I = |R'|^2 P_0 = \frac{u^2}{1+u^2} P_0 \,, \qquad P_{II} = \frac{P_0}{1+u^2} \tag{183}$$

and

$$P_I - P_{II} = \frac{u^2-1}{u^2+1} \quad , \quad 2u\sqrt{P_I P_{II}} = 2u\sqrt{\frac{u^2}{1+u^2} \times \frac{1}{1+u^2}} = \frac{2u^2}{1+u^2} \tag{184}$$

We see that $P_I - P_{II} < \frac{1}{2}(2u\sqrt{P_I P_{II}})$ and, as the wave propagates along z, P_I decreases, P_{II} increases, $P_I - P_{II}$ decreases and $2u\sqrt{P_I P_{II}}$ increases. If we neglect $P_I - P_{II}$ compared to $2u\sqrt{P_I P_{II}}$ in (175) and (176), the two equations become:

$$\frac{dP_I}{dz} = -2uD\sqrt{P_I P_{II}} \tag{185}$$

$$\frac{dP_{II}}{dz} = 2uD\sqrt{P_I P_{II}} \tag{186}$$

With the initial condition $P_I = P_0$ and $P_{II} = 0$ at $z = 0$, the solution of (185) and (186) is:

$$P_I = P_0 \cos^2 uDz, \tag{187}$$

$$P_{II} = P_0 \sin^2 uDz; \tag{188}$$

this corresponds to:

$$|A(z)| = \sqrt{P_0} \cos u \, Dz,$$

$$\text{or} \quad A(z) = A(0) \cos u \, Dz, \tag{189}$$

and, remembering that B leads A by $\frac{3\pi}{2}$, we have:

$$B(z) = -iA(0) \sin u \, Dz, \tag{190}$$

The factor $uD = \dfrac{K_y}{2\beta b} \times \dfrac{u}{1+u^2}$, with $u = \dfrac{K_y^2 + \gamma_5^2}{2K_y \gamma_5} \, \text{sh}\gamma_5 c$; for a sufficient large separation distance we have $u \to \infty$, and therefore $u \simeq \dfrac{K_y^2 + \gamma_5^2}{2K_y \gamma_5} e^{\gamma_5 c}$ and

$$uD \simeq \frac{K_y}{2\beta b} \cdot \frac{1}{u} \simeq \frac{2K_y^2 \gamma_5 e^{-\gamma_5 c}}{(K_y^2 + \gamma_5^2)\beta b} \quad ;$$

thus, we see that this factor is the same coupling coefficient C of eq. (5´). Finally, we observe that the approximate solution (189) - (190) is the Miller-Marcatili solution, and that it is obtained by neglecting the term $P_I - P_{II}$ in our differential equations. Assuming that the factor k used by Miller is a purely imaginary quantity, and observing that setting $1 - R´ = t$ is the same as neglecting the term $P_I - P_{II}$, we conclude that Miller's a-proximation is not accurate except for a very large separation distance between the two guides, in which case Miller's treatment approaches our more rigorous formulation.

4.4.2 Exact Solution.

We introduce a new function Z(z) given by:

$$Z(z) = \frac{\sqrt{P_I P_{II}}}{P_I - P_{II}} \quad , \tag{191}$$

then:

$$\frac{dZ}{dz} = \frac{P_I \frac{dP_{II}}{dz} + P_{II}\frac{dP_I}{dz}}{2\sqrt{P_I P_{II}}(P_I - P_{II})} - \frac{\sqrt{P_I P_{II}}(\frac{dP_I}{dz} - \frac{dP_{II}}{dz})}{(P_I - P_{II})^2} = \frac{1}{2\sqrt{P_I P_{II}}} \cdot \frac{dP_{II}}{dz} + \frac{2\sqrt{P_I P_{II}}}{(P_I - P_{II})^2} \cdot \frac{dP_{II}}{dz}$$

or:

$$\frac{dZ}{dz} = \left(\frac{1}{2Z} + 2Z\right)\frac{1}{P_I - P_{II}} \cdot \frac{dP_{II}}{dz} \quad . \tag{192}$$

Using (176), (191) and (192), we find:

$$\frac{dZ}{dz} = D\left(\frac{1}{2Z} + 2Z\right)(1 + 2uZ) \tag{193}$$

or

$$\frac{2ZdZ}{(1 + 4Z^2)(1 + 2uZ)} = Ddz \tag{194}$$

The left-hand side of (194) can be separated into three terms, and (194) becomes:

$$\frac{4ZdZ}{1 + 4Z^2} + \frac{2udZ}{1 + 4Z^2} - \frac{2udZ}{1 + 2uZ} = 2D(1 + u^2)dz \tag{195}$$

By integration:

$$Ln \frac{\sqrt{1 + 4Z^2}}{1 + 2uZ} = 2D(1 + u^2)z - u \arctan 2Z + Ln G \tag{196}$$

where G is the integration constant. Eq. (196) can be rewritten as:

$$\frac{\sqrt{1+4Z^2}}{1+2uZ} = G\, e^{2D(1+u^2)z - u\arctan 2Z} \qquad ; \tag{197}$$

substituting (191) in (197) and considering that $D = \dfrac{K_y}{2\beta b(1+u^2)}$, we find:

$$P_I - P_{II} + 2u\sqrt{P_I P_{II}} = \frac{P_o}{G}\, e^{-\frac{K_y}{\beta b}z + u\arctan\frac{2\sqrt{P_I P_{II}}}{P_I - P_{II}}} \quad . \tag{198}$$

With the initial conditions $P_I(0) = P_0$, $P_{II}(0) = 0$, we find that $G = 1$ and, finally:

$$P_I - P_{II} + 2u\sqrt{P_I P_{II}} = P_o\, e^{-\frac{K_y}{\beta b} + 2u\arctan\sqrt{\frac{P_{II}}{P_I}}} \quad . \tag{199}$$

Eq. (199) can be rewritten as:

$$P_I = P_o \cos^2\left\{\frac{1}{2u}\left[\frac{K_y}{\beta b}z + Ln\,\frac{P_I - P_{II} + 2u\sqrt{P_I P_{II}}}{P_o}\right]\right\} \tag{200}$$

$$P_{II} = P_o \sin^2\left\{\frac{1}{2u}\left[\frac{K_y}{\beta b}z + Ln\,\frac{P_I - P_{II} + 2u\sqrt{P_I P_{II}}}{P_o}\right]\right\} \tag{201}$$

4.5 Discussion of Solution.

Eqs. (200) and (201) show that P_{II} increases toward a maximum of $\frac{P_0}{2}\left(1 + \frac{u}{\sqrt{1+u^2}}\right)$, and P_I decreases toward a minmum of $\frac{P_0}{2}\left(1 - \frac{u}{\sqrt{1+u^2}}\right)$, where P_{II} reaches its maximum. P_I reaches its minimum when $(z \to \infty)$: this incorrect result is a consequence of the fact that (175) and (176) correspond to increasing P_{II} and decreasing P_I.

According to relations (181) and (182), we must switch to equations (178) and (179) when P_{II} reaches the value $P_0 \dfrac{u^2}{1+u^2}$; this value is smaller than $\dfrac{P_0}{2}(1+\dfrac{u}{\sqrt{1+u^2}})$.

Also, we know that when P_{II} increases, $\dfrac{dP_{II}}{dz}$ decreases; one bounce before the switch takes place we have that $P_I \geq \dfrac{P_0}{1+u^2}$ and $P_{II} \leq \dfrac{P_0 u^2}{1+u^2}$, the values of P_I and P_{II} for which $\dfrac{dP_{II}}{dz}$ is minimum. Substituting in (176) we find:

$$\frac{dP_{II}}{dz}\bigg|_{min} = D\left[\frac{P_0}{1+u^2} - \frac{P_0 u^2}{1+u^2} + 2u\frac{u}{1+u^2}\right] = P_0 D = \frac{P_0}{1+u^2} \cdot \frac{K_y}{2\beta b} \tag{202}$$

The correct solution is therefore obtained in the following manner. $P_I(z)$ and $P_{II}(z)$ must follow formulas (200) and (201) until $\dfrac{dP_{II}}{dz} = \dfrac{P_0}{1+u^2}$ per bounce length; for the next bounce, we must switch to equations (178) and (179), whose solutions are obtained from (200) and (201) by interchanging P_I and P_{II}. When the switch occurs, P_{II} is greater than $\dfrac{P_0 u^2}{1+u^2}$ and P_I is smaller than $\dfrac{P_0}{1+u^2}$.

Our solution is shown in Fig. 6, where the ratio $P_{II}(z)/P_0$ is plotted as a function of z for different values of the parameter u. In the same figure we also plotted the Miller-Marcatili solution, as given by:

$$P_{II} = P_0 \cos^2 Cz \tag{203}$$

with

$$C = \frac{K_y}{2\beta b} \cdot \frac{u}{1+u^2} \quad , \quad u = \frac{K_y^2 + \gamma_s^2}{2K_y \gamma_s} \, Sh\, \gamma_s C \tag{204}$$

We see that the Miller-Marcatili solution approaches ours as u increases, i.e. as c → ∞. In Fig. 6 the curves have been stopped before P_{II} begins to decrease.

According to our solution, there cannot be complete power transfer from guide I to guide II. The maximum power transfer possible increases with the
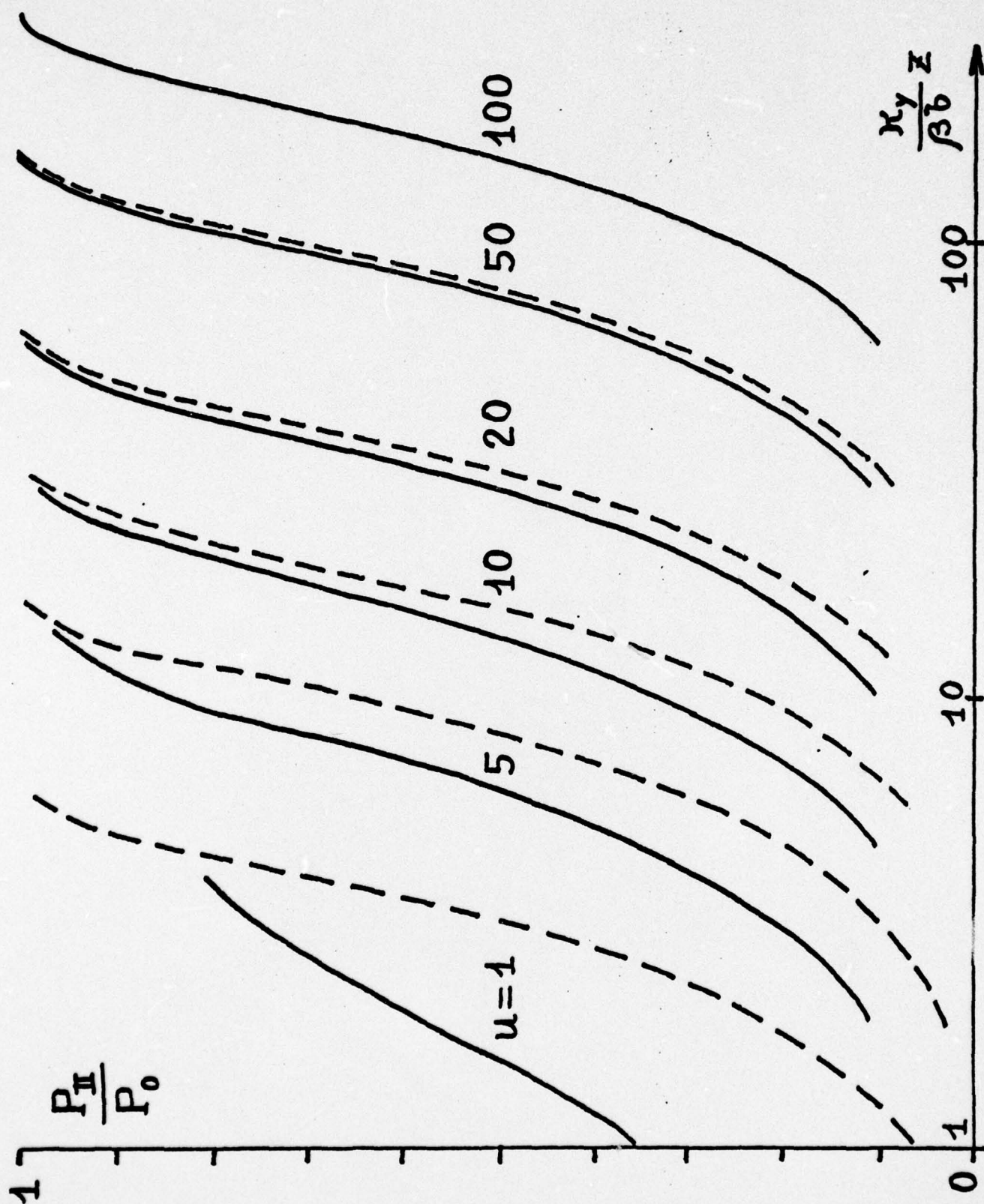
Figure 6. Normalized power in guide II as a function of the coupling length.
——————— Our exact solution. ------------ Marcatili's approximation.

separation distance c, and it reaches 100 percent in the limit c = ∞.

For c = 0, the two guides form one guide with width 2b, then u = 0 and from (176) we find that

$$P_{II} = \frac{1}{2} P_0 \left(1 - \bar{e}^{\frac{\kappa_y}{\beta b} z} \right) , \tag{205}$$

which means that after a few bounce lengths from z = 0, almost half the incident power is transferred to guide II. The maximum transfer of power to guide II is $\frac{P_0}{2}$. In fact, the powers in guides I and II should be equal after the first bounce. We do not obtain this result because our solution is valid only for a small amount of power transfer per bounce.

CONCLUSION

In our procedure for solving the coupling between dielectric optical waveguides we have used three steps:

STEP 1: Calculate the reflection and transmission coefficient at the waveguide boundaries. We saw that phase and magnitude of these two coefficients are functions of the separation distance between the guides.

STEP 2: Solve the eigenvalue equation for modes in the region ($0 \leq z \leq L$). We showed that for not too small values of the separation distance, the incident mode in guide I in the region ($z \leq 0$) couples into a single mode in the region ($0 \leq z \leq L$), and the two modes have the same propagation constant. With this condition, we may neglect reflections at the boundaries ($z = 0$) and ($z = L$).

STEP 3: Derive the differential equations describing the power transfer between the two guides in the most general case. All the coefficients in these equations have been explicitly calculated. Because of the fact that the transmission coefficient between the two guides leads the reflection coefficient by $\frac{3\pi}{2}$, the phase difference between the fields in the two waveguides is a constant along z, and equals $\pm \frac{3\pi}{2}$. With this condition, the solution of the differential equations for the power transfer has been easily obtained. However, the form of the solution is not simple. We saw that there are two different solutions, one valid when the power P_{II} in guide II is increasing, the second when P_{II} is decreasing. However, rules describing how to switch from the first solution to the second were found; this switch occurs when P_{II} reaches its maximum. We showed that there is complete transfer of power only when the separation distance tends to infinity (extremely weak coupling).

Although our theory was developed for the particular case of rectangular waveguides, it could be extended to waveguides of different cross sections. For example, it could even be used to describe the coupling of two optical fibers via a

coaxial dielectric sleeve; this specific problem will be studied in a forthcoming paper.

Finally, we observe that our theory allows us to use the directional coupler as a mode-selecting device. If several modes are incident in guide I, we can choose the separation distance c and the coupling length L to maximize the power transfer to guide II for a given mode only.

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